

AN APPLICATION OF TOPOLOGICAL EQUIVALENCE TO MORSE THEORY

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ABSTRACT. In a previous paper, under the assumption that the Riemannian metric is special, the author proved some results about the moduli spaces and CW structures arising from Morse theory. By virtue of topological equivalence, this paper extends those results by dropping the assumption on the metric.

In particular, we give a strong solution to the following classical question: *Does a Morse function on a compact Riemannian manifold gives rise to a CW decomposition that is homeomorphic to the manifold?*

1. INTRODUCTION

In a previous paper [21], the author proved some results on moduli spaces and CW structures arising from Morse theory in the CF case. By the CF case, we mean the Morse function satisfies the Palais-Smale Condition (C) on a complete Hilbert-Riemannian manifold and its critical points have finite indices (see [21, def. 2.6]). Those results include the manifold structure of the compactified moduli spaces, orientation formulas, and the CW structure on the underlying manifold. (See [21] for a detailed description and a bibliography.)

Most results in [21] are based on the assumption of that the Riemannian metric (or the negative gradient vector field) is locally trivial (see Definition 2.3). This means the vector field has the simplest form near each critical point.

In this paper, by virtue of topological equivalence (see Definition 2.8), we shall extend those results by dropping the above assumption provided that the Morse function is proper. Here the underlying manifold has to be finite dimensional but not necessarily compact.

In order to apply topological equivalence, based on the idea outlined in the paper by Newhouse and Peixoto [16], we shall prove the following main theorem stated in Franks' paper [7, prop. 1.6].

Theorem A. *Suppose f is a Morse function on a compact manifold M . Suppose X is a negative gradient-like field for f (see Definition 2.1), and X satisfies transversality (see Definition 2.5). Then there is a regular path between X and Y such that Y is also a negative gradient-like field for f . More importantly, Y is locally trivial. In particular, there is a topological equivalence between X and Y .*

In Theorem A, by a regular path, we mean a continuous path of negative gradient-like vector fields in which each single vector field on the path satisfies transversality. A precise version of Theorem A is Theorem 4.1.

Key words and phrases. Morse theory, negative gradient-like dynamical system, topological equivalence, Moduli space, compactification, orientation formula, CW structure.

The importance of Theorem A is that it can be combined with the results of [21] to give an extension of those results to more general metrics. In particular, we give a strong solution to the following classical question which had been considered by Thom ([25]), Bott ([2, p. 104]) and Smale ([24, p. 197]): *Does a Morse function on a compact Riemannian manifold give rise to a CW decomposition that is homeomorphic to the manifold such that its open cells are the unstable manifolds of the negative gradient vector field?* A corollary of Theorem 9.1 gives the following answer which strengthens the work in [11] and [12] (see also Remark 9.1):

Theorem B. *Suppose f , M and X are the same as those in Theorem A. Then the compactified unstable manifolds of X give a CW decomposition that is homeomorphic to M . The open cells of this CW complex are the unstable manifolds. Furthermore, the characteristic maps have explicit formulas.*

The following is the reason for making the extension of results in [21]. There are at least two disadvantages of the locally trivial metric assumed in [21]. Firstly, local triviality is not a generic property. Sometimes, especially in the infinite dimensional setting such as in Floer theory, it is not usually the case that one can find a metric satisfying both the local triviality and transversality conditions. Secondly, the assumption of local triviality of the metric contradicts symmetry. Take for example a homogeneous Riemannian manifold. If the metric is locally trivial, then the curvature tensor must vanish near each critical point. Since the metric is homogeneous, the curvature tensor must vanish globally. Thus only a tiny class of homogeneous Riemannian manifolds have this type of metric.

Actually, the local triviality assumption on the metric was made in [21] exclusively because of the techniques employed there. The theorems in [21, thm. 3.3, 3.4, and 3.5] show that, under the assumption of a locally trivial metric, the compactified moduli spaces have smooth structures compatible with that of the underlying manifold. However, the example in [21, example 3.1] shows that, if the metric is not locally trivial, there is no such compatibility (see also Remark 7.2). Thus the case of a locally trivial metric has several distinct features from the general case. In fact, the proofs of [21, thm. 3.7 and 3.8] rely heavily on the compatibility.

In this situation, it's natural to pose the following strategy for obtaining results about Morse moduli spaces in the case of a general metric. As a first step, we implement the subtle and technical arguments in the special case. In the second and final step, we try to convert the general case to the special case. The paper [21] completes the first step. This paper achieves the second one.

Franks' paper [7, prop. 1.6] proposes an excellent idea to reduce the general case to the special case as follows. The proof of [16, lem. 2] claims that there exists a regular path (i.e. each single vector field on the path satisfies transversality) as the one stated in Theorem A. Since a negative gradient-like vector field satisfying transversality is structurally stable, we get the topological equivalence in Theorem A, which converts the general vector field X to the locally trivial Y . (The argument in [7] also shows the power of Theorem A.)

However, there is a serious issue in the proof in [16]. It's well known that, for negative gradient-like vector fields, transversality is preserved under small C^1 perturbations. However, the vector fields certainly change largely in the C^1 topology along the above path. *How can we guarantee the transversality?* Franks' paper [7] refers the proof to [16], and the latter outlines the construction of the path. Both [7]

and [16] indicate that the λ -Lemma in [18] verifies the transversality. Unfortunately, none of them explain *why* the λ -Lemma works in this setting.

The current paper supports the above idea in [16]. Precisely, following this idea, we shall give a self-contained and detailed proof of Theorem 4.1. However, the statement of Theorem 4.1 is slightly different from that in [16] such that it becomes better in the setting of Morse theory. (Actually, the papers [16] and [7] emphasize the setting of dynamical systems. However, our argument also proves the result in [16]. See Remark 4.2.)

The main body of this paper consists of two parts. The first part, Sections 3-5, consists of preparations for the application of topological equivalence. The main theorems in it are Theorems 3.1 and 4.1, which may be of independent interest. The second part consists of the subsequent sections and gives the application of topological equivalence. Theorem 6.7 shows that the compactified moduli spaces are invariants of topological equivalence, which is the base for our application. The theorems in Sections 7-9 are extensions of those in [21].

2. PRELIMINARIES

In this section, we give some definitions, notation and elementary results mostly used throughout the paper.

Suppose M is a finite dimensional smooth manifold, and f is a proper Morse function on M . Denote $f^{-1}([a, b])$ by $M^{a,b}$. Denote $f^{-1}((-\infty, a])$ by M^a .

Definition 2.1. A vector X is a negative gradient-like field for f if $Xf(x) < 0$ when x is not a critical point, and, near each critical point p , X is the negative gradient of f for some metric.

By Definition 2.1, every gradient vector field is obviously a gradient-like vector field. On the contrary, Smale [23, remark after thm. B] gives the following fact (see also [21, lem. 7.12]).

Lemma 2.2. *Every negative gradient-like field of a Morse function f is actually a negative gradient field of f for some metric.*

By the Morse Lemma, there exists a local coordinate chart near a critical point p such that p has the coordinate $(0, 0)$, and the function has the form

$$(2.1) \quad f(x_1, x_2) = f(p) - \frac{1}{2}\langle x_1, x_1 \rangle + \frac{1}{2}\langle x_2, x_2 \rangle$$

in this chart. We call this chart a Morse chart.

Definition 2.3. We say the metric of M is trivial near p if the metric of M coincides with the standard metric of a Morse chart near p . In other words, in this Morse chart, $-\nabla f$ has the simplest form $-\nabla f(x_1, x_2) = (x_1, -x_2)$. Similarly, we say a negative gradient-like field X is trivial near p if $X(x_1, x_2) = (x_1, -x_2)$ in a Morse chart. If the metric (or X) is trivial near each critical point, we say this metric (or X) is locally trivial.

Remark 2.1. Some papers in the literature include the local triviality of X into the definition of a gradient-like vector field. We follow the style of [23] and exclude it.

Definition 2.4. Let $\phi_t(x)$ be the flow generated by X with initial value x . Suppose p is a critical point. Define the descending manifold of p as $\mathcal{D}(p) = \{x \in$

$M \mid \lim_{t \rightarrow -\infty} \phi_t(x) = p\}$. Define the ascending manifold of p as $\mathcal{A}(p) = \{x \in M \mid \lim_{t \rightarrow +\infty} \phi_t(x) = p\}$. We call $\mathcal{D}(p)$ and $\mathcal{A}(p)$ the invariant manifolds of p . We also denote $\mathcal{D}(p)$ by $\mathcal{D}(p; X)$ and denote $\mathcal{A}(p)$ by $\mathcal{A}(p; X)$ in order to indicate the vector field X .

Clearly, $\mathcal{D}(p)$ is the unstable manifold of p with respect to X , and $\mathcal{A}(p)$ is the stable manifold. They are smoothly embedded open disks in M . Furthermore, $\dim(\mathcal{D}(p)) = \text{ind}(p)$, where $\text{ind}(p)$ is the Morse index of p .

Definition 2.5. We say that X satisfies transversality if $\mathcal{D}(p)$ and $\mathcal{A}(q)$ are transversal for all critical points p and q . For critical points p and q , we say that p and q are transversal if the invariant manifolds of p are transverse to those of q . Suppose U is a subset of M , and these invariant manifolds meet transversally at each point in U (this includes the case that they don't meet at that point). We say that p and q are transversal in U .

The following lemma is obvious.

Lemma 2.6. *If p and q are transversal in $f^{-1}((a, b))$ and $p \in f^{-1}((a, b))$, then p and q are transversal. If p and q are transversal in $f^{-1}(a)$ and $f(q) < a < f(p)$, then p and q are transversal.*

Definition 2.7. Suppose p and q are critical points. Define $q \preceq p$ if there exists a flow from p to q . Define $q \prec p$ if $q \preceq p$ and $q \neq p$.

If X satisfies transversality, then “ \preceq ” is a partial order on the set consisting of all critical points (see [19, p. 85, cor. 1]).

Now we introduce the definitions of topological conjugacy and topological equivalence in dynamical systems. The reader is to be forewarned that the definitions appearing the literature are not uniform. We follow the terminology of [19, p. 26]. In this paper, a topological conjugacy is a relation strictly stronger than a topological equivalence. This is *different* from the definition in [7]. The “topological conjugacy” in [7, p. 201] is actually the “topological equivalence” in this paper. Although a topological equivalence is good enough for our application to Morse theory, we still introduce the notion of topological conjugacy in order to make the statement of Theorem 4.1 stronger.

Definition 2.8. Suppose X_i ($i = 1, 2$) is a vector field on M_i and ϕ_t^i is the flow generated by X_i . Suppose $h : M_1 \rightarrow M_2$ is a homeomorphism. If $h\phi_t^1 = \phi_t^2 h$, then we call h a topological conjugacy between X_1 and X_2 . If h maps the orbits of X_1 to the orbits of X_2 and h preserves the directions of orbits, then we call h is a topological equivalence between X_1 and X_2 .

Remark 2.2. In dynamical systems, people usually consider the topological equivalence (or conjugacy) of vector fields on one manifold M , i.e. $M_1 = M_2$ in Definition 2.8. However, it seems beneficial for topology to allow that M_1 is not diffeomorphic to M_2 . For example, choose a standard sphere S^n and an exotic sphere Σ^n . Let f_1 and f_2 be the height functions on S^n and Σ^n respectively. We can define a topological conjugacy between $-\nabla f_1$ and $-\nabla f_2$ as follows. Choose a homeomorphism (or even a diffeomorphism) $h_0 : S^{n-1} \rightarrow \Sigma^{n-1}$, where S^{n-1} and Σ^{n-1} are the equators of S^n and Σ^n respectively. Define h such that $h\phi_t^1(x) = \phi_t^2 h_0(x)$ for all $x \in S^{n-1}$, and h maps the maximum (minimum) point to the maximum (minimum) point. Clearly, this topological conjugacy h recovers the Alexander trick.

3. A STRENGTHENED MORSE LEMMA

In this section, we shall present a Strengthened Morse Lemma which is useful for the proof of Theorem 4.1 (See Remarks 4.1 and 4.2).

Suppose H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and U is an open subset of H . Define a smooth Riemannian metric (or smooth metric for brevity) on U in the usual sense. In other words, for each $x \in U$, assign a symmetric positive definite linear operator $A(x)$ such that $A(x)$ is a smooth function of x . For any v and w in $T_x U = H$, define $\langle v, w \rangle_{G(x)} = \langle A(x)v, w \rangle$.

Theorem 3.1 (Strengthened Morse Lemma). *Suppose H is a Hilbert space, U is an open neighborhood of $0 \in H$. Suppose f is a smooth Morse function on U with a critical point 0 , and G is a smooth metric on U . Let $-\nabla_G f$ be the negative gradient of f with respect to G , and ϕ_t be the flow generated by $-\nabla_G f$. Suppose $H = H_1 \oplus H_2$, where H_1 and H_2 are the negative and positive spectral spaces of $\nabla_G^2 f(0)$ respectively. Then there exist an open neighborhood V of 0 such that $V \subseteq U$, $B_1 = \{x_1 \in H_1 \mid \|x_1\| < \epsilon\}$, $B_2 = \{x_2 \in H_2 \mid \|x_2\| < \epsilon\}$, and a diffeomorphism $h : B_1 \times B_2 \rightarrow V$ such that the following holds. We have*

$$(3.1) \quad \begin{aligned} h^* f(x_1, x_2) &= f(0) - \frac{1}{2} \langle x_1, x_1 \rangle + \frac{1}{2} \langle x_2, x_2 \rangle, \\ h(B_1) &= \mathcal{D}_V(0; -\nabla_G f) = \{x \in V \mid \phi((-\infty, 0], x) \subseteq V\} \\ &= \left\{ x \in V \mid \phi((-\infty, 0], x) \subseteq V, \lim_{t \rightarrow -\infty} \phi(t, x) = 0 \right\}, \end{aligned}$$

and

$$\begin{aligned} h(B_2) &= \mathcal{A}_V(0; -\nabla_G f) = \{x \in V \mid \phi([0, +\infty), x) \subseteq V\} \\ &= \left\{ x \in V \mid \phi([0, +\infty), x) \subseteq V, \lim_{t \rightarrow +\infty} \phi(t, x) = 0 \right\}. \end{aligned}$$

Before proving it, we explain the statement of Theorem 3.1. In this theorem, $\mathcal{D}_V(0; -\nabla_G f)$ is the local unstable (descending) manifold of 0 in the neighborhood V , and $\mathcal{A}_V(0; -\nabla_G f)$ is the local stable (ascending) manifold. They certainly depend on the metric. The classical Morse Lemma shows that, by a coordinate transformation h , we get a new chart (we call it a Morse Chart) such that the function has the form (3.1) in it. Theorem 3.1 tells us more: No matter what the metric is, there exists a Morse chart such that the local invariant manifolds are standard in it. (Figure 1 illustrates this strengthened Morse chart, where the arrows indicate the directions of the flows.) This makes three objects, i.e. the function, the local invariant manifolds, and the coordinate chart fit well. In short, Theorem 3.1 strengthens the classical Morse Lemma by taking the dynamical system into account.

Proof of Theorem 3.1. We know that ϕ_1 is a smooth map defined on U_0 with a hyperbolic fixed point 0 , where U_0 is a neighborhood of 0 . By the Local Invariant Manifold Theorem (see [8] and [9, thm. 28]), shrinking U_0 suitably, there exists a diffeomorphism $h_1 : \tilde{B}_1 \times \tilde{B}_2 \rightarrow U_0$ such that

$$\begin{aligned} h_1(\tilde{B}_1) &= \mathcal{D}_{U_0}(0; \phi_1) = \{x \in U_0 \mid \forall n \leq 0, (\phi_1)^n(x) \in U_0\} \\ &= \left\{ x \in U_0 \mid \forall n \leq 0, (\phi_1)^n(x) \in U_0, \lim_{n \rightarrow -\infty} (\phi_1)^n(x) = 0 \right\}, \end{aligned}$$

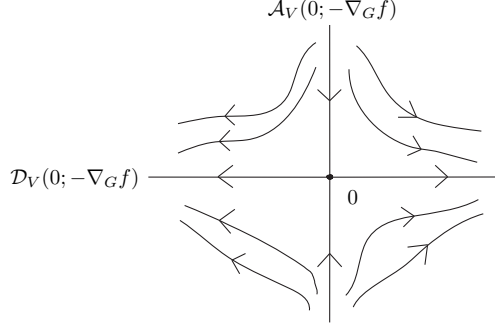


FIGURE 1. Strengthened Morse Chart

and $h_1(\tilde{B}_2) = \mathcal{A}_{U_0}(0; \phi_1)$. Here the definition of $\mathcal{A}_{U_0}(0; \phi_1)$ is similar to that of $\mathcal{D}_{U_0}(0; \phi_1)$, and $(0, 0) \in \tilde{B}_1 \times \tilde{B}_2 \subseteq H_1 \times H_2$.

Clearly, $h_1^*f|_{\tilde{B}_1}$ and $h_1^*f|_{\tilde{B}_2}$ are Morse functions on \tilde{B}_1 and \tilde{B}_2 respectively. By the Morse Lemma, composing h_1 with a diffeomorphism if necessary, we may assume that

$$h_1^*f|_{\tilde{B}_1} = f(0) - \frac{1}{2}\langle x_1, x_1 \rangle, \quad h_1^*f|_{\tilde{B}_2} = f(0) + \frac{1}{2}\langle x_2, x_2 \rangle.$$

Define

$$R(x) = h_1^*f(x) - \left(f(0) - \frac{1}{2}\langle x_1, x_1 \rangle + \frac{1}{2}\langle x_2, x_2 \rangle \right).$$

Here $x = (x_1, x_2)$. Denote the differential of R with order n by $D^n R$. Then $R(x_1, 0) \equiv 0$ and $R(0, x_2) \equiv 0$. In addition, for any $v_1 \in H_1$ and $v_2 \in H_2$, we have

$$\begin{aligned} D^2(h_1^*f)(0)(v_1, v_2) &= D^2f(Dh_1 \cdot v_1, Dh_1 \cdot v_2) \\ &= \langle \nabla_G^2 f(0) Dh_1 \cdot v_1, Dh_1 \cdot v_2 \rangle_{G(0)}. \end{aligned}$$

We know that $Dh_1 \cdot v_1 \in H_1$, $Dh_1 \cdot v_2 \in H_2$, $\nabla_G^2 f(0)$ is symmetric with respect to $G(0)$, and H_1 and H_2 are negative and positive spectral spaces of $\nabla_G^2 f(0)$ respectively. Thus $D^2(h_1^*f)(0)(v_1, v_2) = 0$. We infer $D^2R(0)(v_1, v_2) = 0$ and $D_{1,2}^2R(0) = 0$.

Now we have

$$\begin{aligned}
& R(x_1, x_2) \\
&= R(x_1, 0) + \int_0^1 \frac{d}{dt} R(x_1, tx_2) dt \\
&= \int_0^1 D_2 R(x_1, tx_2) dt \cdot x_2 \quad (\text{because } R(x_1, 0) = 0) \\
&= \int_0^1 \left[D_2 R(0, tx_2) + \int_0^1 \frac{d}{ds} D_2 R(sx_1, tx_2) ds \right] dt \cdot x_2 \\
&= \int_0^1 \int_0^1 D_{1,2}^2 R(sx_1, tx_2) ds dt (x_1, x_2) \quad (\text{because } D_2 R(0, tx_2) = 0) \\
&= \int_0^1 \int_0^1 \left[D_{1,2}^2 R(0, 0) + \int_0^1 \frac{d}{d\tau} D_{1,2}^2 R(\tau sx_1, \tau tx_2) d\tau \right] ds dt (x_1, x_2) \\
&= \int_0^1 \int_0^1 \int_0^1 s D_{1,1,2}^3 R(\tau sx_1, \tau tx_2) d\tau ds dt (x_1, x_1, x_2) \quad (\text{because } D_{1,2}^2 R(0) = 0) \\
&\quad + \int_0^1 \int_0^1 \int_0^1 t D_{1,2,2}^3 R(\tau sx_1, \tau tx_2) d\tau ds dt (x_1, x_2, x_2).
\end{aligned}$$

Since $D^3 R$ is a symmetric multilinear form, there exists symmetric operators $R_1(x)$ and $R_2(x)$ on H_1 and H_2 respectively such that, for any v_1 and w_1 in H_1 ,

$$\int_0^1 \int_0^1 \int_0^1 s D_{1,1,2}^3 R(\tau sx_1, \tau tx_2) d\tau ds dt (v_1, w_1, x_2) = \frac{1}{2} \langle R_1(x_1, x_2) v_1, w_1 \rangle;$$

and, for any v_2 and w_2 in H_2 ,

$$\int_0^1 \int_0^1 \int_0^1 t D_{1,2,2}^3 R(\tau sx_1, \tau tx_2) d\tau ds dt (x_1, v_2, w_2) = \frac{1}{2} \langle R_2(x_1, x_2) v_2, w_2 \rangle.$$

Here $R_1(x)$ and $R_2(x)$ are smooth with respect to x .

Clearly, $R_1(0) = 0$, $R_2(0) = 0$, and

$$h_1^* f(x) = f(0) - \frac{1}{2} \langle (I - R_1)(x) x_1, x_1 \rangle + \frac{1}{2} \langle (I + R_2)(x) x_2, x_2 \rangle.$$

Since $I - R_1$ and $I + R_2$ are symmetric, $I - R_1(0) = I$ and $I + R_2(0) = I$, shrinking $\tilde{B}_1 \times \tilde{B}_2$ if necessary, we have $I - R_1(x) = C_1(x)^2$ and $I + R_2(x) = C_2(x)^2$. Here $C_1(x)$ and $C_2(x)$ are symmetric and positive definite operators on H_1 and H_2 respectively, and they are smooth functions of x . Thus

$$h_1^* f(x) = f(0) - \frac{1}{2} \langle C_1(x) x_1, C_1(x) x_1 \rangle + \frac{1}{2} \langle C_2(x) x_2, C_2(x) x_2 \rangle.$$

Define $h_2 : \tilde{B}_1 \times \tilde{B}_2 \rightarrow H_1 \times H_2$ by $h_2(x) = (C_1(x) x_1, C_2(x) x_2)$. Then $h_2(\tilde{B}_1) \subseteq H_1$ and $h_2(\tilde{B}_2) \subseteq H_2$. Since $Dh_2(0) = I$, there exists $\hat{B}_1 \times \hat{B}_2 \subseteq H_1 \times H_2$ such that h_2^{-1} exists and is smooth on $\hat{B}_1 \times \hat{B}_2$. Then we get

$$(h_2^{-1} \circ h_1)^* f(x) = f(0) - \frac{1}{2} \langle x_1, x_1 \rangle + \frac{1}{2} \langle x_2, x_2 \rangle.$$

Define $B_1 = \{x_1 \in H_1 \mid \|x_1\| < \epsilon\} \subseteq h_1^{-1}(\hat{B}_1)$, $B_2 = \{x_2 \in H_2 \mid \|x_2\| < \epsilon\} \subseteq h_1^{-1}(\hat{B}_2)$, $h = h_2^{-1} \circ h_1$ and $V = h(B_1 \times B_2)$.

We see that $h^{-1}(\mathcal{D}_{U_0}(0; \phi_1)) = B_1$ and $h^{-1}(\mathcal{A}_{U_0}(0; \phi_1)) = B_2$. By the fact that $-\nabla_G f \cdot f \leq 0$, it is straightforward to prove that $h(B_1) = \mathcal{D}_V(0; -\nabla_G f)$ and $h(B_2) = \mathcal{A}_V(0; -\nabla_G f)$. \square

4. A REGULAR PATH

As mentioned in the Introduction, the purpose of this section is to present a detailed proof of Theorem 4.1 in order to support an idea outlined in [16, lem. 2]. In this proof, Lemma 4.3 plays a key role.

Theorem 4.1 (Regular Path). *Suppose f is a Morse function on a compact manifold M . Suppose X is a negative gradient-like field for f , and X satisfies transversality. Then there is a continuous path $\mathcal{Y} : [0, 1] \rightarrow \mathfrak{X}^\infty(M)$ such that, for all $s \in [0, 1]$, \mathcal{Y}_s is a negative gradient-like field for f , \mathcal{Y}_s satisfies transversality, $\mathcal{Y}_0 = X$ and \mathcal{Y}_1 is locally trivial. In particular, there exists a topological conjugacy h between X and \mathcal{Y}_1 such that $h(p) = p$ for each critical point p . Here $\mathfrak{X}^\infty(M)$ is the set with the Whitney C^∞ topology consisting of C^∞ vector fields on M .*

We call a continuous path of negative gradient-like vector fields $\mathcal{Y} : [a, b] \rightarrow \mathfrak{X}^\infty(M)$ a regular path if \mathcal{Y}_s satisfies transversality for all s .

We need the following classical Comparison Theorem for ODEs (see [26, p. 96]).

Theorem 4.2 (well-known). *Suppose $F(t, x)$ is a Lipschitz continuous function defined on $[t_0, t_1] \times [a, b]$. Let $x(t)$ be the solution of the equation $\dot{x} = F(t, x)$ with $x(t_0) = x_0$. Suppose $y(t)$ is a C^1 function defined on $[t_0, t_1]$ with $y(t_0) = x_0$. Then*

- (1) *if $\dot{y} \leq F(t, y)$, then $y(t) \leq x(t)$ on $[t_0, t_1]$;*
- (2) *if $\dot{y} \geq F(t, y)$, then $y(t) \geq x(t)$ on $[t_0, t_1]$.*

Suppose $H = H_1 \oplus H_2$ is a Hilbert space, $v = (v_1, v_2) \in H$, $v_1 \neq 0$, and $\lambda = \frac{\|v_2\|}{\|v_1\|}$. We call λ the inclination of v with respect to H_1 . Suppose L is a closed subspace of H , and $P : H \rightarrow H_1$ is the projection. If $P : L \rightarrow P(L)$ is a topological linear isomorphism, then there exists a bounded linear operator $A : P(L) \rightarrow H_2$ such that L is the graph of A , i.e., for any $v \in L$, we have $v = (v_1, Av_1)$, where $v_1 \in P(L)$. We call the supremum of the inclinations of all non-zero vectors in L the inclination of L with respect to H_1 . Clearly, the inclination of L equals, $\|A\|$, the norm of A .

Suppose H , H_1 and H_2 are Hilbert spaces as above. Suppose A_0 and A_1 are linear operators on H_1 , and B is a linear operator on H_2 . There exist positive numbers $\alpha_0 > 0$, $\alpha_1 > 0$ and $\beta > 0$ such that

$$(4.1) \quad \alpha_0 \langle w, w \rangle \leq \langle A_i w, w \rangle \leq \alpha_1 \langle w, w \rangle \quad (i = 0, 1),$$

and

$$(4.2) \quad \beta \langle w, w \rangle \leq \langle Bw, w \rangle.$$

Let ρ be a smooth bump function on $(-\infty, +\infty)$ such that $0 \leq \rho \leq 1$, $\rho(s) \equiv 1$ when $s \leq \frac{1}{2}$, and $\rho(s) \equiv 0$ when $s \geq 1$. Define $\rho_r(s) = \rho(\frac{s}{r})$ for $r > 0$. For convenience, we denote $\rho_r(\|x_i\|)$ by $\rho_r(x_i)$, where $x_i \in H_i$.

Define a smooth vector field X_r on H by

$$X_r(x_1, x_2) = (\rho_r(x_1)\rho_r(x_2)A_0x_1 + [1 - \rho_r(x_1)\rho_r(x_2)]A_1x_1, -Bx_2).$$

Denote the flow generated by X_r by $\phi_t(x_1, x_2)$. For a fixed t , ϕ_t is a diffeomorphism, thus $D\phi_t$ acts on the tangent vectors at each point (x_1, x_2) , where $D\phi_t$ is the differential of ϕ_t with respect to $x = (x_1, x_2)$.

Lemma 4.3. *For any $\epsilon > 0$, there exists $\delta > 0$ such that the following holds. For any $r > 0$ and $v \in H$, if the inclination of v with respect to H_1 is less than δ , then we have the inclination of $D\phi_t \cdot v$ with respect to H_1 is less than ϵ for all $t \geq 0$. Here δ only depends on $\alpha_0, \alpha_1, \beta$ and ϵ , and δ is independent of r .*

Proof. The flow $\phi_t = (\phi_t^1, \phi_t^2)$ satisfies the following ordinary differential equation

$$\begin{cases} \dot{\phi}^1 &= \rho_r(\phi^1)\rho_r(\phi^2)A_0\phi^1 + [1 - \rho_r(\phi^1)\rho_r(\phi^2)]A_1\phi^1, \\ \dot{\phi}^2 &= -B\phi^2. \end{cases}$$

Denote $\rho_r(\phi^1)\rho_r(\phi^2)A_0 + [1 - \rho_r(\phi^1)\rho_r(\phi^2)]A_1$ by $A(\phi^1, \phi^2)$. We have

$$\frac{d}{dt}\langle \phi^1, \phi^1 \rangle = 2\langle \dot{\phi}^1, \phi^1 \rangle = 2\langle A(\phi^1, \phi^2)\phi^1, \phi^1 \rangle.$$

By (4.1), we have

$$0 \leq 2\alpha_0\langle \phi^1, \phi^1 \rangle \leq \frac{d}{dt}\langle \phi^1, \phi^1 \rangle \leq 2\alpha_1\langle \phi^1, \phi^1 \rangle.$$

Thus $\|\phi^1\|$ is increasing, and by Theorem 4.2, we have

$$(4.3) \quad e^{\alpha_0 t}\|\phi_0^1\| \leq \|\phi_t^1\| \leq e^{\alpha_1 t}\|\phi_0^1\|.$$

Similarly, $\|\phi^2\|$ is decreasing, $\phi_t^2 = e^{-Bt}\phi_0^2$, and $\|\phi_t^2\| \leq e^{-\beta t}\|\phi_0^2\|$.

Let $\mathbb{D}_1(r) = \{x_1 \in H_1 \mid \|x_1\| < r\}$, and $\mathbb{D}_2(r) = \{x_2 \in H_2 \mid \|x_2\| < r\}$. Clearly, $A(x_1, x_2)|_{H - (\mathbb{D}_1(r) \times \mathbb{D}_2(r))} = A_1$, and $A(x_1, x_2)|_{\overline{\mathbb{D}_1(\frac{r}{2}) \times \mathbb{D}_2(\frac{r}{2})}} = A_0$. Denote $\overline{\mathbb{D}_1(r) \times \mathbb{D}_2(r)} - (\mathbb{D}_1(\frac{r}{2}) \times \mathbb{D}_2(\frac{r}{2}))$ by $E(r)$. When $\phi([0, t], x)$ is out of $E(r)$, we have $\phi(t, x) = (e^{A_1 t}x_1, e^{-Bt}x_2)$, and

$$D\phi_t = \begin{pmatrix} e^{A_1 t} & 0 \\ 0 & e^{-Bt} \end{pmatrix}.$$

Since $\|e^{A_1 t}w\| \geq \|w\|$ and $\|e^{-Bt}w\| \leq \|w\|$ for $t \geq 0$, we have that the inclination of $D\phi_t \cdot v$ is decreasing when t is increasing. Thus it suffices to control the variation of the inclination when $\phi_t(x)$ passes through $E(r)$.

Suppose $t \geq 0$ and $\|\phi_t^1\| = 2\|\phi_0^1\|$, then by (4.3), we have $t \leq \frac{\ln 2}{\alpha_0}$. Similarly, if $\|\phi_t^2\| = \frac{1}{2}\|\phi_0^2\|$, then $t \leq \frac{\ln 2}{\beta}$. Since $\|\phi_t^1\|$ is increasing and $\|\phi_t^2\|$ is decreasing, we infer that ϕ_t enters $E(r)$ at most twice, and the time for it to stay in $E(r)$ is no more than

$$(4.4) \quad T = \frac{\ln 2}{\alpha_0} + \frac{\ln 2}{\beta}.$$

Suppose $\phi([0, t], x) \subset E(r)$, we have $0 \leq t \leq T$. Since $\phi_t^2(x) = e^{-Bt}x_2$, we have

$$(4.5) \quad D_1\phi_t^2 = 0, \quad D_2\phi_t^2 = e^{-Bt}, \quad \text{and} \quad \|D_2\phi_t^2 \cdot w\| \leq \|w\|.$$

Since

$$\dot{\phi}^1 = A(\phi^1, e^{-Bt}x_2)\phi^1,$$

we have

$$D_1\dot{\phi}^1 \cdot w = A(\phi^1, \phi^2)(D_1\phi^1 \cdot w) + D\rho_r(\phi^1)(D_1\phi^1 \cdot w)\rho_r(\phi^2)(A_0 - A_1)\phi^1.$$

Thus

$$\begin{aligned} \frac{d}{dt} \langle D_1 \phi^1 \cdot w, D_1 \phi^1 \cdot w \rangle &= 2 \langle D_1 \dot{\phi}^1 \cdot w, D_1 \phi^1 \cdot w \rangle \\ &= 2 \langle A(\phi^1, \phi^2)(D_1 \phi^1 \cdot w), D_1 \phi^1 \cdot w \rangle \\ &\quad + 2 \langle D\rho_r(\phi^1)(D_1 \phi^1 \cdot w) \rho_r(\phi^2)(A_0 - A_1)\phi^1, D_1 \phi^1 \cdot w \rangle. \end{aligned}$$

Clearly, $D\rho_r(\phi^1) = O(r^{-1})$, and $\|\phi^1\| \leq r$ when $D\rho_r(\phi^1) \neq 0$. So there exists a constant $C_1 > 0$ which is independent of r such that

$$|\langle D\rho_r(\phi^1)(D_1 \phi^1 \cdot w) \rho_r(\phi^2)(A_0 - A_1)\phi^1, D_1 \phi^1 \cdot w \rangle| \leq C_1 \|D_1 \phi^1 \cdot w\|^2.$$

Combining the above inequality with (4.1), we get

$$-2C_1 \langle D_1 \phi^1 \cdot w, D_1 \phi^1 \cdot w \rangle \leq \frac{d}{dt} \langle D_1 \phi^1 \cdot w, D_1 \phi^1 \cdot w \rangle.$$

Since $D_1 \phi_0^1 = I$ and $\|D_1 \phi_0^1 \cdot w\| = \|w\|$, by Theorem 4.2, we have

$$(4.6) \quad \|D_1 \phi_t^1 \cdot w\| \geq e^{-C_1 t} \|w\| \geq e^{-C_1 T} \|w\|.$$

Similarly, we have

$$\begin{aligned} \frac{d}{dt} \langle D_2 \phi^1 \cdot w, D_2 \phi^1 \cdot w \rangle &= 2 \langle A(\phi^1, \phi^2)(D_2 \phi^1 \cdot w), D_2 \phi^1 \cdot w \rangle \\ &\quad + 2 \langle D\rho_r(\phi^1)(D_2 \phi^1 \cdot w) \rho_r(\phi^2)(A_0 - A_1)\phi^1, D_2 \phi^1 \cdot w \rangle \\ &\quad + 2 \langle \rho_r(\phi^1) D\rho_r(\phi^2) e^{-Bt} w (A_0 - A_1)\phi^1, D_2 \phi^1 \cdot w \rangle, \end{aligned}$$

and

$$|\langle D\rho_r(\phi^1)(D_2 \phi^1 \cdot w) \rho_r(\phi^2)(A_0 - A_1)\phi^1, D_2 \phi^1 \cdot w \rangle| \leq C_1 \|D_2 \phi^1 \cdot w\|^2.$$

In addition, $\rho_r(\phi^1) D\rho_r(\phi^2) = O(r^{-1})$, and $\|\phi^1\| \leq r$ when $\rho_r(\phi^1) D\rho_r(\phi^1) \neq 0$. So there exists $C_2 > 0$ which is independent of r such that

$$\begin{aligned} &2 |\langle \rho_r(\phi^1) D\rho_r(\phi^2) e^{-Bt} w (A_0 - A_1)\phi^1, D_2 \phi^1 \cdot w \rangle| \\ &\leq 2C_2 \|D_2 \phi^1 \cdot w\| \|w\| \leq C_2 \|D_2 \phi^1 \cdot w\|^2 + C_2 \|w\|^2. \end{aligned}$$

Thus by (4.1), we infer

$$\frac{d}{dt} \langle D_2 \phi^1 \cdot w, D_2 \phi^1 \cdot w \rangle \leq (2\alpha_1 + 2C_1 + C_2) \langle D_2 \phi^1 \cdot w, D_2 \phi^1 \cdot w \rangle + C_2 \|w\|^2.$$

Since $\|D_2 \phi_0^1 \cdot w\| = 0$, by Theorem 4.2 again, there exists a $C_3 > 0$ which is independent of r such that

$$(4.7) \quad \|D_2 \phi_t^1 \cdot w\| \leq \left[\frac{C_2}{C_3} (e^{C_3 T} - 1) \right]^{\frac{1}{2}} \|w\|.$$

By (4.4), (4.6) and (4.7), there exist $K_1 > 0$ and $K_2 > 0$, which are independent of r , such that

$$(4.8) \quad \|D_1 \phi_t^1 \cdot w\| \geq K_1 \|w\|, \quad \text{and} \quad \|D_2 \phi_t^1 \cdot w\| \leq K_2 \|w\|.$$

Suppose $v = (v_1, v_2) \in H_1 \oplus H_2$, and its inclination is $\lambda_0 = \frac{\|v_2\|}{\|v_1\|}$. By (4.5) and (4.8), we have the inclination of $D\phi_t^1 \cdot v$ is

$$\begin{aligned} \lambda_1 &= \frac{\|D_2 \phi_t^2 \cdot v_2\|}{\|D_1 \phi_t^1 \cdot v_1 + D_2 \phi_t^1 \cdot v_2\|} \leq \frac{\|D_2 \phi_t^2 \cdot v_2\|}{\|D_1 \phi_t^1 \cdot v_1\| - \|D_2 \phi_t^1 \cdot v_2\|} \\ &\leq \frac{\|v_2\|}{K_1 \|v_1\| - K_2 \|v_2\|} = \frac{\lambda_0}{K_1 - K_2 \lambda_0}. \end{aligned}$$

Thus λ_1 tends to 0 when λ_0 tends to 0.

Since $\phi_t(x)$ enters $E(r)$ at most twice, and K_1 and K_2 are independent of r , the proof is completed. \square

The following definition of *filtration* is a special case of that in hyperbolic dynamical systems (see [17, p. 1029]).

Definition 4.4. A compact submanifold M_1 with boundary inside M is a filtration for X if $\dim(M_1) = \dim(M)$, $\phi_t(M_1) \subseteq \text{Int}M_1$ for $t > 0$, and X is transverse to ∂M_1 . Here $\text{Int}M_1$ is the interior of M_1 , and ϕ_t is the flow generated by X .

Lemma 4.5. *Suppose X satisfies transversality. If p and q are critical points such that $p \not\preceq q$, then there exists a filtration M_1 such that $p \in M - M_1$ and $q \in \text{Int}M_1$.*

Lemma 4.5 can be proved as follows. The transversality implies “ \preceq ” is a partial order. We have $p \not\preceq q_1$ if $q_1 \preceq q$. Using [14, thm. 4.1] repeatedly, we can modify f to be a Morse function g such that X is a negative gradient-like field for g and $g(q) < g(p)$. The proof is finished.

By Definition 2.1, we have the following obvious lemma.

Lemma 4.6. *Suppose X_1 and X_2 are negative gradient-like fields of f . Suppose $\sigma_1(x)$ and $\sigma_2(x)$ are nonnegative smooth functions on M such that $\sigma_1 + \sigma_2 > 0$. Then $\sigma_1 X_1 + \sigma_2 X_2$ is also a negative gradient-like field for f .*

Let p be a critical point. Suppose there exists a Morse chart near p (see (2.1)), and $X(x_1, x_2) = (Ax_1, -Bx_2)$, where A and B are symmetric positive definite linear operators. Similarly to Lemma 4.3, define

$$\mathcal{Y}_r(x_1, x_2) = (\rho_r(x_1)\rho_r(x_2)x_1 + [1 - \rho_r(x_1)\rho_r(x_2)]Ax_1, -Bx_2)$$

in this Morse chart and $\mathcal{Y}_r = X$ out of this Morse chart. For $s \in [0, 1]$, define

$$\mathcal{Y}_{r,s} = (1 - s)X + s\mathcal{Y}_r.$$

By Lemma 4.6, for all $s \in [0, 1]$, $\mathcal{Y}_{r,s}$ is a negative gradient-like field for f .

Lemma 4.7. *Suppose X satisfies transversality. Then when r is small enough, we have the following conclusion.*

Suppose q_1 and q_2 are two critical points which are not of the following two cases: (1) $q_2 \prec p \prec q_1$; or (2) $q_1 \prec p \prec q_2$. Then we have that q_1 and q_2 are transversal with respect to $\mathcal{Y}_{r,s}$ for all $s \in [0, 1]$. Here “ \prec ” is defined with respect to X .

Proof. Clearly, $\mathcal{Y}_{r,s}$ differs from X only in a neighborhood U_r of p . When r tends to 0, U_r shrinks to p .

We may assume that $f(q) \neq f(p)$ for any critical point q such that $q \neq p$. If this is not true, perturb f to be a Morse function \tilde{f} such that X is a negative gradient-like field for \tilde{f} , and $\tilde{f}(x) = f(x) + C$ in a neighborhood U of p . Let r be small enough such that $U_r \subseteq U$. Then $\mathcal{Y}_{r,s}$ is also a negative gradient-like field for \tilde{f} . For the rest of the proof we make the above assumption.

Suppose $U_r \subseteq M^{a,b}$ and p is the unique singularity in $M^{a,b}$. As in Definition 2.4, we use notation $\mathcal{D}(*; *)$ and $\mathcal{A}(*; *)$ to indicate the vector fields.

It's easy to see that $\mathcal{D}(p; \mathcal{Y}_{r,s}) = \mathcal{D}(p; X)$. Suppose that $q \in M^a$. Since $\mathcal{Y}_{r,s}$ is identical to X in $M - M^{a,b}$, we have $\mathcal{A}(q; \mathcal{Y}_{r,s}) \cap M^a = \mathcal{A}(q; X) \cap M^a$. Since X satisfies transversality, we infer that p and q are transversal in M^a with respect to $\mathcal{Y}_{r,s}$. By Lemma 2.6, p and q are transversal globally. Similarly, if $q \in M - M^a$, p

and q are also transversal. As a result, p and q are transversal. It suffices to check the case that $q_1 \neq p$ and $q_2 \neq p$.

If $p \not\prec q$, by Lemma 4.5, there exists a filtration M_1 such that $q \in \text{Int}M_1$ and $p \in M - M_1$. Let r be small enough such that $U_r \subseteq M - M_1$, then $\mathcal{Y}_{r,s}$ is identical to X on M_1 . So $\mathcal{D}(q; \mathcal{Y}_{r,s}) = \mathcal{D}(q; X)$. Similarly, if $q \not\prec p$, we can get $\mathcal{A}(q; \mathcal{Y}_{r,s}) = \mathcal{A}(q; X)$ when r is small enough. Thus there exists $r_0 > 0$ such that the following holds. When $r < r_0$, we have, for all $s \in [0, 1]$, $\mathcal{D}(q; \mathcal{Y}_{r,s}) = \mathcal{D}(q; X)$ if $p \not\prec q$, and $\mathcal{A}(q; \mathcal{Y}_{r,s}) = \mathcal{A}(q; X)$ if $q \not\prec p$.

In order to complete this proof, we only need to check the following three cases.

(1). Case 1: q_1 and q_2 are in M^a .

Since $\mathcal{Y}_{r,s}$ is identical to X on M^a and X satisfies transversality, we have q_1 and q_2 are transversal in M^a . By Lemma 2.6, they are transversal globally.

(2). Case 2: q_1 and q_2 are in $M - M^a$.

Similarly to Case (1), this case is also true.

(3). Case 3: one of q_1 and q_2 is in $M - M^a$ and the other one is in M^a .

We may presume $q_1 \in M - M^a$ and $q_2 \in M^a$. By the assumption of this lemma, we have either $p \not\prec q_1$ or $q_2 \not\prec p$. Suppose $p \not\prec q_1$. We have $\mathcal{D}(q_1; \mathcal{Y}_{r,s}) = \mathcal{D}(q_1; X)$. Since X satisfies transversality, we have q_1 and q_2 are transversal in M^a with respect to $\mathcal{Y}_{r,s}$. By Lemma 2.6, they are transversal globally. Similarly, if $q_2 \not\prec p$, this is also true. Thus Case 3 is also verified. \square

We shall strengthen Lemma 4.7 to get the transversality of $\mathcal{Y}_{r,s}$. Recall a classical result on transversality at first.

Suppose U is a neighborhood of p such that U is identified with a neighborhood of 0 in $T_p M = H_1 \oplus H_2$, and p is identified with 0, where $H_1 = T_p \mathcal{D}(p; X)$ and $H_2 = T_p \mathcal{A}(p; X)$. Furthermore, suppose $\mathcal{D}(p; X) \cap U \subseteq H_1$ and $\mathcal{A}(p; X) \cap U \subseteq H_2$. Then we have the following crucial fact: When U is small enough, there exists $\Lambda > 0$ such that for any $q_1 \succeq p$ and any $x \in \mathcal{D}(q_1; X) \cap U$, there exists a linear space $V_x^d \subseteq T_x \mathcal{D}(q_1; X)$ such that $\dim(V_x^d) = \dim(H_1)$ and the inclination of V_x^d with respect to H_1 is less than Λ . Similarly, for any $q_2 \preceq p$ and any $x \in \mathcal{A}(q_2; X) \cap U$, there exists $V_x^a \subseteq T_x \mathcal{A}(q_2; X)$ such that $\dim(V_x^a) = \dim(H_2)$ and the inclination of V_x^a with respect to H_2 is also less than Λ . In addition, Λ tends to 0 when U shrinks to p . This fact follows from the transversality of X and the estimate of the λ -Lemma. (Note: the λ -Lemma is also named the Inclination Lemma.) On the contrary, we assume this fact holds but do not assume the transversality of X . If $\Lambda < 1$, then, for any $x \in \mathcal{D}(q_1; X) \cap \mathcal{A}(q_2; X) \cap U$, we have

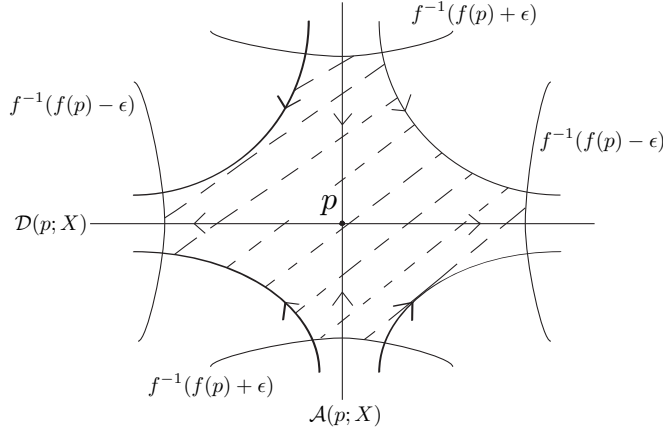
$$T_x M = H_1 \oplus H_2 = V_x^d \oplus V_x^a = T_x \mathcal{D}(q_1; X) + T_x \mathcal{A}(q_2; X).$$

So we infer that $\mathcal{D}(q_1; X)$ and $\mathcal{A}(q_2; X)$ are transversal in U . The above argument is the key part of the proof of that, for Morse-Smale dynamical systems, transversality is preserved under small C^1 perturbations. All of these are addressed in [18, lem. 1.11 and thm. 3.5]. In the proof of the following lemma, we shall apply a similar argument to large C^1 perturbations of X .

Lemma 4.8. *Suppose X satisfies transversality. When r is small enough, we have $\mathcal{Y}_{r,s}$ satisfies transversality for all $s \in [0, 1]$.*

Proof. By Lemma 4.7, it suffices to prove that $\mathcal{D}(q_1; \mathcal{Y}_{r,s})$ is transverse to $\mathcal{A}(q_2; \mathcal{Y}_{r,s})$ if $q_2 \prec p \prec q_1$.

Similarly to the proof of Lemma 4.7, we assume that p is the unique critical point in $M^{f(p)-\epsilon, f(p)+\epsilon}$. Let U be the neighborhood of p in the argument before this lemma. Let D be an open subset of $f^{-1}(f(p)+\epsilon) \cap U$ such that $D \supseteq f^{-1}(f(p)+\epsilon) \cap \mathcal{A}(p; X)$. Let $U_0 = [\phi([0, +\infty), D) \cup \mathcal{D}(p; X)] \cap M^{f(p)-\epsilon, f(p)+\epsilon}$. Then U_0 is a neighborhood of p and is relatively open in $M^{f(p)-\epsilon, f(p)+\epsilon}$. When ϵ tends to 0 and D shrinks, U_0 shrinks to p . (In Figure 2, the shadowed part is U_0 , the arrows indicate the the directions of the flows.) Denote the flow generated by $\mathcal{Y}_{r,s}$ by $\phi_t^{r,s}$.


 FIGURE 2. Neighborhood U_0

Both $M^{f(p)-\epsilon, f(p)+\epsilon} - U_0$ and U_0 are unions of some complete orbits generated by X in $M^{f(p)-\epsilon, f(p)+\epsilon}$. Let U_0 be small enough such that $U_0 \subseteq U$. Choose $U_1 \subseteq U_0$ such that U_1 is also a union of some complete orbits generated by X in $M^{f(p)-\epsilon, f(p)+\epsilon}$, and U_1 is a closed neighborhood of p . Let r be small enough such that $\mathcal{Y}_{r,s}$ is identical to X out of U_1 . We have $M^{f(p)-\epsilon, f(p)+\epsilon} - U_0$ is still the union of some complete orbits generated by $\mathcal{Y}_{r,s}$ in $M^{f(p)-\epsilon, f(p)+\epsilon}$. Then so is U_0 . Thus, for any $x \in [f^{-1}(f(p) + \epsilon) \cap U_0] - \mathcal{A}(p; X)$, we have $\phi^{r,s}(t, x) \in f^{-1}(f(p) - \epsilon)$ for some $t > 0$ and $\phi^{r,s}([0, t]) \subset U_0$.

We know that

$$\mathcal{Y}_{r,s}(x_1, x_2) = (\rho_r(x_1)\rho_r(x_2)(sI + (1-s)A)x_1 + [1 - \rho_r(x_1)\rho_r(x_2)]Ax_1, -Bx_2),$$

and there exist $\alpha_0 > 0$, $\alpha_1 > 0$ and $\beta > 0$ such that, for any $s \in [0, 1]$, we have

$$\alpha_0 I \leq sI + (1-s)A \leq \alpha_1 I, \quad \alpha_0 I \leq A \leq \alpha_1 I, \quad \text{and} \quad \beta I \leq B.$$

By Lemma 4.3, there exists $\delta > 0$ such that the following holds. Suppose $x \in \mathcal{D}(q_1; X) \cap f^{-1}(f(p) + \epsilon) \cap U_0$, and $V_x^d \subseteq T_x \mathcal{D}(q_1; X)$ is the space described before this lemma. If the inclination of V_x^d with respect to H_1 is less than δ , then, in U_0 , the inclination of $\phi_t^{r,s}(V_x^d)$ with respect to H_1 is less than 1. It's necessary to point out that δ is independent of r and s .

Clearly, $\mathcal{D}(q_1; X) \cap f^{-1}([f(p) + \epsilon, +\infty)) = \mathcal{D}(q_1; \mathcal{Y}_{r,s}) \cap f^{-1}([f(p) + \epsilon, +\infty))$ and $\mathcal{A}(q_2; X) \cap M^{f(p)-\epsilon} = \mathcal{A}(q_2; \mathcal{Y}_{r,s}) \cap M^{f(p)-\epsilon}$. Since X satisfies transversality, by the argument before this lemma, we can choose U_0 be small enough such that the following holds. For any $x \in \mathcal{D}(q_1; X) \cap f^{-1}(f(p) + \epsilon) \cap U_0$, the inclination of V_x^d

with respect to H_1 is less than δ , and, for any $y \in \mathcal{A}(q_2; X) \cap f^{-1}(f(p) - \epsilon) \cap U_0$, the inclination of V_y^a with respect to H_2 is less than 1. Here $V_x^d \subseteq T_x \mathcal{D}(q_1; X) = T_x \mathcal{D}(q_1; \mathcal{Y}_{r,s})$ and $V_y^a \subseteq T_y \mathcal{A}(q_2; X) = T_y \mathcal{A}(q_2; \mathcal{Y}_{r,s})$. Thus, if $\phi_t^{r,s}(x) = y$, then the inclination of $V_y^d = D\phi_t^{r,s} \cdot V_x^d$ with respect to H_1 is less than 1. Here $V_y^d \subseteq T_y \mathcal{D}(q_1; \mathcal{Y}_{r,s})$. By the argument before this lemma again, we have $T_y M = V_y^d \oplus V_y^a$. So $\mathcal{D}(q_1; \mathcal{Y}_{r,s})$ and $\mathcal{A}(q_2; \mathcal{Y}_{r,s})$ are transversal in $f^{-1}(f(p) - \epsilon) \cap U_0$.

Furthermore, $\mathcal{D}(q_1; X) \cap (M^{f(p)-\epsilon, f(p)+\epsilon} - U_1) = \mathcal{D}(q_1; \mathcal{Y}_{r,s}) \cap (M^{f(p)-\epsilon, f(p)+\epsilon} - U_1)$ and $\mathcal{A}(q_2; X) \cap (M^{f(p)-\epsilon, f(p)+\epsilon} - U_1) = \mathcal{A}(q_2; \mathcal{Y}_{r,s}) \cap (M^{f(p)-\epsilon, f(p)+\epsilon} - U_1)$. Thus $\mathcal{D}(q_1; \mathcal{Y}_{r,s})$ and $\mathcal{A}(q_2; \mathcal{Y}_{r,s})$ are transversal in $M^{f(p)-\epsilon, f(p)+\epsilon} - U_1$.

In summary, $\mathcal{D}(q_1; \mathcal{Y}_{r,s})$ and $\mathcal{A}(q_2; \mathcal{Y}_{r,s})$ are transversal in $f^{-1}(f(p) - \epsilon)$. By Lemma 2.6, they are transversal globally. \square

Proof of Theorem 4.1. First, we construct the regular path. It suffices to prove that, for any critical point p , we can construct a regular path \mathcal{Y} such that $\mathcal{Y}_0 = X$ and \mathcal{Y}_1 is locally trivial at p .

By Theorem 3.1, there exists a coordinate chart U near p such that p has coordinate $(0, 0)$,

$$f(x_1, x_2) = f(p) - \frac{1}{2}\langle x_1, x_1 \rangle + \frac{1}{2}\langle x_2, x_2 \rangle,$$

$\mathcal{D}(p; X) \cap U = \{(x_1, 0)\}$ and $\mathcal{A}(p; X) \cap U = \{(0, x_2)\}$. Clearly, $D^2 X(p) = (A, -B)$, where A and B are symmetric and positive definite. Furthermore, $(Ax_1, -Bx_2)$ is also a gradient-like vector field for f near p .

Let ρ_r be the bump function defined before. For convenience, for all $x = (x_1, x_2)$, denote $\rho_r(\|x\|)$ by $\rho_r(x)$. Let $R(x) = X(x) - (Ax_1, -Bx_2)$. Then we have $\|\rho_r(x)R(x)\|$ and $\|D[\rho_r(x)R(x)]\|$ tend to 0 when r tends to 0. Since the transversality of X is preserved under small C^1 perturbations, we have $Z_s = X - s\rho_r R$ is a regular path when r is small enough and $s \in [0, 1]$. Clearly, $Z_1(x) = (Ax_1, -Bx_2)$ near p . By Lemma 4.8, we can construct a regular path $Z_{[1,2]}$ such that $Z_2(x) = (x_1, -Bx_2)$ near p . Since $-Z_2$ is a negative gradient-like field for $-f$, using Lemma 4.8 again, we can construct a regular path $Z_{[2,3]}$ such that $Z_3(x) = (x_1, -x_2)$ near p . We get the desired path by defining $\mathcal{Y}_s = Z_{3s}$.

Second, we prove the existence of the conjugacy h .

By the proof in [20, thm. 5.2], we know that, for each \mathcal{Y}_{s_0} , there is a topological equivalence h_{s_0} between \mathcal{Y}_{s_0} and \mathcal{Y}_s such that $h_{s_0}(p) = p$ for all critical points p when s is close to s_0 enough. In addition, since the flow generated by \mathcal{Y}_{s_0} has no closed orbits, by the comment in [20, p. 231], we know that h_{s_0} is actually a conjugacy. Thus it's easy to get the desired conjugacy h . \square

Remark 4.1. In the proof of Theorem 4.1, we need to choose a Morse chart $U \subseteq H_1 \oplus H_2$ such that H_1 and H_2 are respectively the tangent spaces of $\mathcal{D}(p; X)$ and $\mathcal{A}(p; X)$ at p . This is *not* granted because these tangent spaces depend on the metric. Theorem 3.1 provides this.

Remark 4.2. The regular path in [16] consists of the Morse-Smale vector fields without closed orbits. In this case, $DX(p) = (A, -B)$ for singularities p , where A and B are linear isomorphisms whose eigenvalues have positive real parts. The paper [16] claims that there exists a regular path connecting X with Y such that $Y(x_1, x_2) = (2x_1, -2x_2)$ near each singularity. Thus, in the setting of dynamical systems, this result is more general than Theorem 4.1. However, Theorem 4.1 has

the advantage that its vector fields are negative gradient-like for f . This is the reason that we need Theorem 3.1. Furthermore, the argument in this paper can also be used to verify the result in [16]. This is because we can choose a metric near each critical point, for example, by the real Jordan canonical form, such that the above operators A and B satisfy (4.1) and (4.2).

5. A REDUCTION LEMMA

In this paper, we shall prove theorems for noncompact manifolds with proper Morse functions. However, the manifold in Theorem 4.1 is required to be compact. The following lemma reduces the proper case to the compact case.

Lemma 5.1. *Suppose M is a compact manifold with boundary $\partial M = M_1 \sqcup M_2$. Here M_i ($i = 1, 2$) may be empty. Suppose f is a Morse function on M such that $f|_{M_1} \equiv a$, $f|_{M_2} \equiv b$, a and b are regular values of f , and $a < b$. Suppose X is a negative gradient-like vector field for f , and X satisfies transversality. Then there exist a compact manifold \widetilde{M} without boundary and a smooth embedding $i : M \hookrightarrow \widetilde{M}$ such that the following holds. There exist a Morse function \widetilde{f} and its negative gradient-like vector field \widetilde{X} on \widetilde{M} . They are extensions of f and X respectively, and \widetilde{X} satisfies transversality. For any critical points p and q in M , we have $\mathcal{D}(p; \widetilde{X}) \cap \mathcal{A}(q; \widetilde{X}) = \mathcal{D}(p; X) \cap \mathcal{A}(q; X)$. Furthermore, $\mathcal{D}(p; \widetilde{X}) = \mathcal{D}(p; X)$ and $\widetilde{f}|_{\widetilde{M}-M} > b$ if $M_1 = \emptyset$; and $\mathcal{A}(p; \widetilde{X}) = \mathcal{A}(p; X)$ and $\widetilde{f}|_{\widetilde{M}-M} < a$ if $M_2 = \emptyset$.*

Proof. If $\partial M = \emptyset$, let $\widetilde{M} = M$, the proof is finished. Now we assume $\partial M \neq \emptyset$.

Let \widetilde{M} be the double of M . Extend f to be \widetilde{f} such that a and b are its regular values, and extend X to be \widetilde{X} which is a negative gradient-like field for \widetilde{f} . (Figure 3 illustrates the manifold \widetilde{M} , where the Morse function is the height function and the shadowed part is M .) We shall modify \widetilde{X} such that it satisfies transversality. The method of such a modification is Milnor's sliding invariant (descending or ascending) manifolds in [14, thm. 5.2]. Basically, there are two ways of sliding invariant manifolds in order to get transversality. Method 1 is sliding the descending manifolds one by one with the order from critical points with lower values to those with higher values. On the contrary, Method 2 is sliding the ascending manifolds one by one with the order from critical points with higher values to those with lower values. Our method is a combination of the above two methods.

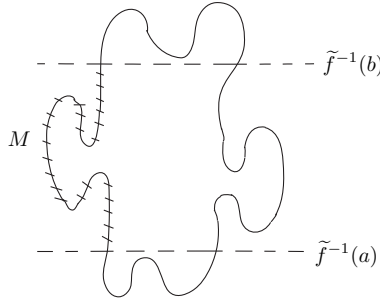


FIGURE 3. Manifold \widetilde{M}

In this proof, we say two critical points \tilde{p} and \tilde{q} of \tilde{f} are transversal if they are transversal with respect to \tilde{X} .

Step 1: we show the transversality between $p \in M$ and $q \in M$. Since $\mathcal{D}(p; \tilde{X}) \subseteq M \cup \text{Int} \tilde{M}^a$, we have $\mathcal{D}(p; \tilde{X}) \cap \tilde{M}^{a,b} = \mathcal{D}(p; \tilde{X}) \cap M = \mathcal{D}(p; X)$. Similarly, $\mathcal{A}(p; \tilde{X}) \cap \tilde{M}^{a,b} = \mathcal{A}(p; X)$. Since X satisfies transversality, p and q are transversal in $\tilde{M}^{a,b}$. By Lemma 2.6, they are transversal globally. This shows the transversality between p and q does not depend on the extension of X . So, no matter how \tilde{X} is changed outside of M , p and q are always transversal if they are in M .

Step 2: we modify \tilde{X} in \tilde{M}^a . We made modifications near each critical point \tilde{p} in \tilde{M}^a with the order from critical points with higher values to those with lower values. Slide $\mathcal{A}(\tilde{p}; \tilde{X})$ for each $\tilde{p} \in \tilde{M}^a$ such that \tilde{p} is transverse to each $\tilde{q} \in M \cup \tilde{M}^a$ with $\tilde{f}(\tilde{q}) \geq \tilde{f}(\tilde{p})$. (Here, for all $\tilde{q} \in M$, we have $\tilde{f}(\tilde{q}) > \tilde{f}(\tilde{p})$.) Thus, for all \tilde{p} and \tilde{q} in $M \cup \tilde{M}^a$, they are transversal globally after these modifications. By Lemma 2.6 and Step 1, no matter how \tilde{X} is changed outside of $M \cup \tilde{M}^a$, \tilde{p} and \tilde{q} are still transversal globally because they are still transversal in \tilde{M}^a .

Step 3: we modify \tilde{X} in $\tilde{M}^b - [M \cup \tilde{M}^a]$. To do this, we slide the descending manifolds with the order from critical points with lower values to those with higher values. More precisely, slide $\mathcal{D}(\tilde{p}; \tilde{X})$ for each $\tilde{p} \in \tilde{M}^b - [M \cup \tilde{M}^a]$ such that \tilde{p} is transverse to all $\tilde{q} \in \tilde{M}^b - M$ with $\tilde{f}(\tilde{q}) < \tilde{f}(\tilde{p})$. (Here, for all $\tilde{q} \in \tilde{M}^a$, we have $\tilde{f}(\tilde{q}) < \tilde{f}(\tilde{p})$.) We claim that, for all \tilde{p} and \tilde{q} in \tilde{M}^b , they are transversal. It suffices to prove that, for each $p \in M$ and $\tilde{q} \in \tilde{M}^{a,b} - M$, we have p and \tilde{q} are transversal. Clearly, $\mathcal{D}(\tilde{q}; \tilde{X}) \subseteq \tilde{M}^b - M$, thus $\mathcal{D}(\tilde{q}; \tilde{X}) \cap \tilde{M}^{a,b} \subseteq \tilde{M}^{a,b} - M$. Since $\mathcal{A}(p; \tilde{X}) \cap \tilde{M}^{a,b} \subseteq M$, we get $\mathcal{D}(\tilde{q}; \tilde{X}) \cap \mathcal{A}(p; \tilde{X}) \cap \tilde{M}^{a,b} = \emptyset$. So $\mathcal{D}(\tilde{q}; \tilde{X}) \cap \mathcal{A}(p; \tilde{X}) = \emptyset$. Similarly, $\mathcal{A}(\tilde{q}; \tilde{X}) \cap \mathcal{D}(p; \tilde{X}) = \emptyset$. We infer that p and \tilde{q} are transversal. The above claim is proved. By Lemma 2.6 again, no matter how \tilde{X} is changed outside of \tilde{M}^b , all critical points in \tilde{M}^b are still mutually transverse.

Step 4: we modify \tilde{X} on $\tilde{M} - \tilde{M}^b$. Slide the descending manifolds with the order from critical points with lower values to those with higher values. We eventually get that \tilde{X} satisfies transversality.

By the above argument, for all p and q in M , we have $\mathcal{D}(p; \tilde{X}) \subseteq M \cup \tilde{f}^{-1}((-\infty, a))$, $\mathcal{A}(q; \tilde{X}) \subseteq M \cup \tilde{f}^{-1}((b, +\infty))$, $\mathcal{D}(p; \tilde{X}) \cap M = \mathcal{D}(p; X)$ and $\mathcal{A}(q; \tilde{X}) \cap M = \mathcal{A}(q; X)$. Thus

$$\mathcal{D}(p; \tilde{X}) \cap \mathcal{A}(q; \tilde{X}) = (\mathcal{D}(p; \tilde{X}) \cap M) \cap (\mathcal{A}(q; \tilde{X}) \cap M) = \mathcal{D}(p; X) \cap \mathcal{A}(q; X).$$

Suppose $M_1 = \emptyset$. Clearly, we can construct \tilde{f} such that $\tilde{f}|_{\tilde{M}-M} > b$. Thus, for any $p \in M$, we have $\mathcal{D}(p; \tilde{X}) \subseteq M$ and $\mathcal{D}(p; \tilde{X}) = \mathcal{D}(p; X)$. Similarly, the conclusion is true in the case of $M_2 = \emptyset$. \square

6. MODULI SPACES AND TOPOLOGICAL EQUIVALENCE

In this section, we shall review the definitions of moduli spaces and their compactifications. These definitions are standard in the literature (see e.g. [3], [4], [5], [13], [22] and [21]). There are several ways to define the topology of these spaces. All of them result in the same topology. The definitions in this paper follow those in [21, thms. 3.3, 3.4 and 3.5].

The paper [21] focuses on the negative *gradient* vector fields. This paper deals with the negative *gradient-like* vector fields. By Lemma 2.2, there is no difference.

After this review, we shall prove Theorem 6.7. This theorem shows that topologically equivalent negative gradient-like fields have homeomorphic compactified moduli spaces. In other words, the compactified moduli spaces are invariants of topological equivalence. In this paper, the application of topological equivalence to Morse theory is based on this theorem.

Let M be a finite dimensional manifold. Let f be a proper Morse function on M and X be a negative gradient-like vector field for f . Assume X satisfies transversality. Denote by $\phi_t(x)$ the flow generated by X with initial value x . Define an equivalence relation on M by

$$x \sim y \iff y = \phi_t(x) \text{ for some } t \in (-\infty, +\infty).$$

Then $x \sim y$ if and only if x and y lie on the same flow line. Suppose p and q are critical points of f . Define $\mathcal{W}(p, q) = \mathcal{D}(p) \cap \mathcal{A}(q)$. Then $\mathcal{W}(p, q)$ is a smoothly embedded submanifold of M . Define $\mathcal{M}(p, q) = \mathcal{W}(p, q) / \sim$. We define the smooth structure of $\mathcal{M}(p, q)$ as follows. Choose a regular value $a \in (f(q), f(p))$. Then each flow line in $\mathcal{W}(p, q)$ intersects $f^{-1}(a)$ exactly at one point. This identifies $\mathcal{M}(p, q)$ with $\mathcal{W}(p, q) \cap f^{-1}(a)$ naturally. We transfer the smooth structure of $\mathcal{W}(p, q) \cap f^{-1}(a)$ to $\mathcal{M}(p, q)$ by this identification. Clearly, this definition does not depend on the choice of a . Furthermore, the natural projection from $\mathcal{W}(p, q)$ to $\mathcal{M}(p, q)$ is a smooth submersion.

It's well known that $\dim(\mathcal{W}(p, q)) = \text{ind}(p) - \text{ind}(q)$ and $\dim(\mathcal{M}(p, q)) = \text{ind}(p) - \text{ind}(q) - 1$.

We shall generalize the concept of flow lines. Suppose γ is a flow line. If it passes through a singularity, it is a constant flow line. Otherwise, it is nonconstant. The following definitions follow [21, sec. 2]

Definition 6.1. An ordered sequence of flow lines $\Gamma = (\gamma_1, \dots, \gamma_n)$, $n \geq 1$, is a generalized flow line if $\gamma_i(+\infty) = \gamma_{i+1}(-\infty)$ and γ_i are constant or nonconstant alternatively according to the order of their places in the sequence. We call γ_i a component of Γ .

Definition 6.2. Suppose x and y are two points in M . A generalized flow line $(\gamma_1, \dots, \gamma_n)$ connects x with y if there exist $t_1, t_2 \in (-\infty, +\infty)$ such that $\gamma_1(t_1) = x$ and $\gamma_n(t_2) = y$. A point z is a point on $(\gamma_1, \dots, \gamma_n)$ if there exists γ_i and $t \in (-\infty, +\infty)$ such that $\gamma_i(t) = z$.

Definition 6.3. An ordered set $I = \{r_0, r_1, \dots, r_{k+1}\}$ is a critical sequence if r_i ($i = 0, \dots, k+1$) are critical points and $r_0 \succ r_1 \succ \dots \succ r_{k+1}$. We call r_0 the head of I , and r_{k+1} the tail of I . The length of I is $|I| = k$.

Suppose $I = \{r_0, r_1, \dots, r_{k+1}\}$ is a critical sequence. Define

$$\mathcal{M}_I = \prod_{i=0}^k \mathcal{M}(r_i, r_{i+1}).$$

Define a space $\overline{\mathcal{M}(p, q)}$ as

$$(6.1) \quad \overline{\mathcal{M}(p, q)} = \bigsqcup_I \mathcal{M}_I,$$

where the disjoint union is over all critical sequence with head p and tail q . As mentioned before, “ \succeq ” is a partial order because of transversality.

We can give $\overline{\mathcal{M}(p, q)}$ another equivalent definition which is sometimes more convenient. If $\alpha \in \mathcal{M}_I \subseteq \overline{\mathcal{M}(p, q)}$, then $\alpha = (\gamma_0, \dots, \gamma_k)$, where $\gamma_i \in \mathcal{M}(r_i, r_{i+1})$, $r_0 = p$ and $r_{k+1} = q$. Denote the constant flow line passing through r_i by $\beta(r_i)$. We can identify α with the generalized flow line $(\beta(r_0), \gamma_0, \beta(r_1), \dots, \gamma_k, \beta(r_{k+1}))$ connecting p with q . Thus we get

$$\overline{\mathcal{M}(p, q)} = \{\Gamma \mid \Gamma \text{ is a generalized flow line connecting } p \text{ with } q\}.$$

Suppose the critical values of f divide $[f(q), f(p)]$ into $l + 1$ intervals $[c_{i+1}, c_i]$ ($i = 0, \dots, l$), where $c_0 = f(p)$ and $c_{l+1} = f(q)$. Choose a regular value $a_i \in (c_{i+1}, c_i)$. The generalized flow line $\Gamma \in \overline{\mathcal{M}(p, q)}$ intersects with $f^{-1}(a_i)$ at exactly one point $x_i(\Gamma)$. There is an evaluation map $E : \overline{\mathcal{M}(p, q)} \rightarrow \prod_{i=0}^l f^{-1}(a_i)$ which is injective and is defined as

$$(6.2) \quad E(\Gamma) = (x_0(\Gamma), \dots, x_l(\Gamma)).$$

Definition 6.4. Define the set $\overline{\mathcal{M}(p, q)}$ as (6.1). Equip $\overline{\mathcal{M}(p, q)}$ with the unique topology such that the evaluation map $E : \overline{\mathcal{M}(p, q)} \rightarrow \prod_{i=0}^l f^{-1}(a_i)$ in (6.2) is a topological embedding. We call $\overline{\mathcal{M}(p, q)}$ the compactified moduli space of $\mathcal{M}(p, q)$.

It's easy to see that the definition of the topology of $\overline{\mathcal{M}(p, q)}$ does not depend on the choice of a_i .

We compactify $\mathcal{W}(p, q)$ to be $\overline{\mathcal{W}(p, q)}$ as follows.

Suppose $I_1 = (p, r_1, \dots, r_s)$ and $I_2 = (r_{s+1}, \dots, r_k, q)$ are critical sequences such that $r_s \succeq r_{s+1}$. Let $(I, s) = (p, r_1, \dots, q)$. Denote $\mathcal{M}_{I_1} \times \mathcal{W}(r_s, r_{s+1}) \times \mathcal{M}_{I_2}$ by $\mathcal{W}_{I, s}$.

Define a space $\overline{\mathcal{W}(p, q)}$ as

$$(6.3) \quad \overline{\mathcal{W}(p, q)} = \bigsqcup_{(I, s)} \mathcal{W}_{I, s},$$

where the disjoint union is over all $(I, s) = (p, r_1, \dots, r_k, q)$ such that $p \succ r_1 \succ \dots \succ r_s \succeq r_{s+1} \succ \dots \succ r_k \succ q$ for all k .

Suppose $(\alpha_1, x, \alpha_2) \in \mathcal{M}_{I_1} \times \mathcal{W}(r_s, r_{s+1}) \times \mathcal{M}_{I_2} = \mathcal{W}_{I, s}$. Then x is on the generalized flow line $\Gamma \in \overline{\mathcal{M}(p, q)}$ such that α_1 and α_2 are components of Γ . Thus, identify (α_1, x, α_2) with (Γ, x) , we get

$$\overline{\mathcal{W}(p, q)} = \{(\Gamma, x) \in \overline{\mathcal{M}(p, q)} \times M \mid \Gamma \in \overline{\mathcal{M}(p, q)}, x \text{ is on } \Gamma\}.$$

Definition 6.5. Define the set $\overline{\mathcal{W}(p, q)}$ as (6.3). Define the topology of $\overline{\mathcal{W}(p, q)}$ as the restriction of that of $\overline{\mathcal{M}(p, q)} \times M$. We call $\overline{\mathcal{W}(p, q)}$ the compactified space of $\mathcal{W}(p, q)$.

Clearly, the map $\tilde{E} : \overline{\mathcal{W}(p, q)} \rightarrow \prod_{i=0}^l f^{-1}(a_i) \times M$ is a topological embedding, where

$$(6.4) \quad \tilde{E}(\Gamma, x) = (E(\Gamma), x).$$

Thus the topology of $\overline{\mathcal{W}(p, q)}$ in this paper is equivalent to that of [21, thm. 3.5].

Finally, we define the compactified space $\overline{\mathcal{D}(p)}$ of $\mathcal{D}(p)$. Suppose f is bounded below.

Suppose $I = \{p, r_1, \dots, r_k\}$ is a critical sequence. Denote $\mathcal{M}_I \times \mathcal{D}(r_k)$ by \mathcal{D}_I .

Define a space $\overline{\mathcal{D}(p)}$ as

$$(6.5) \quad \overline{\mathcal{D}(p)} = \bigsqcup_I \mathcal{D}_I,$$

where the disjoint union is over all critical sequences with head p .

Suppose $(\alpha, x) \in \mathcal{M}_I \times \mathcal{D}(r_k) = \mathcal{D}_I$. We can identify α with a generalized flow line connecting p with r_k . Adding the flow line passing through x to the above generalized flow line, we get a generalized flow line connecting p with x . Thus we get

$$\overline{\mathcal{D}(p)} = \{(\Gamma, x) \mid \Gamma \text{ is a generalized flow line connecting } p \text{ with } x\}.$$

The definition of the topology of $\overline{\mathcal{D}(p)}$ is slightly complicated.

Suppose the critical values in $(-\infty, f(p)]$ are exactly $c_l < \dots < c_0 = f(p)$. Define $U(i) \subseteq \overline{\mathcal{D}(p)}$ ($i = 0, \dots, l$) as

$$(6.6) \quad U(i) = \{(\Gamma, x) \mid c_{i+1} < f(x) < c_{i-1}\},$$

where $c_{l+1} = -\infty$ and $c_{-1} = +\infty$. Clearly, $\overline{\mathcal{D}(p)} = \bigcup_i U(i)$. We have the following injection $E_i : U(i) \rightarrow \prod_{j=0}^{i-1} f^{-1}(a_j) \times M$ such that

$$E_i(\Gamma, x) = (x_0(\Gamma), \dots, x_{i-1}(\Gamma), x),$$

where $x_j(\Gamma)$ is the unique intersection point between Γ and $f^{-1}(a_j)$. Equip $U(i)$ the unique topology such that E_i is a topological embedding. The paper [21, thm. 3.4] shows that these $U(i)$ have compatible smooth structures under the assumption of the local triviality of the vector field. Follow that argument, we can prove that the topologies of these $U(i)$ are compatible even if we drop the local triviality. This means that $U(i)$ and $U(j)$ share the same topology on $U(i) \cap U(j)$.

Definition 6.6. Define the set $\overline{\mathcal{D}(p)}$ as (6.5). Define the topology of $\overline{\mathcal{D}(p)} = \bigcup_i U(i)$ as the coherent topology such that each $U(i)$ is an open subspace of $\overline{\mathcal{D}(p)}$ (see (6.6)). We call $\overline{\mathcal{D}(p)}$ the compactified space of $\mathcal{D}(p)$.

Suppose f_1 and f_2 are Morse functions on M_1 and M_2 . Suppose X_i is a negative gradient-like field for f_i , and X_i satisfies transversality. Suppose $h : M_1 \rightarrow M_2$ is a topological equivalence between X_1 and X_2 . If p is a critical point of f_1 , then $h(p)$ is a critical point of f_2 . Furthermore, $h(\mathcal{D}(p)) = \mathcal{D}(h(p))$, $h(\mathcal{A}(p)) = \mathcal{A}(h(p))$, and $h(\mathcal{W}(p, q)) = \mathcal{W}(h(p), h(q))$. Thus h naturally induces maps $h_* : \overline{\mathcal{M}(p, q)} \rightarrow \overline{\mathcal{M}(h(p), h(q))}$, $h_* : \overline{\mathcal{W}(p, q)} \rightarrow \overline{\mathcal{W}(h(p), h(q))}$, and $h_* : \overline{\mathcal{D}(p)} \rightarrow \overline{\mathcal{D}(h(p))}$. Here, if $\Gamma \in \overline{\mathcal{M}(h(p), h(q))}$, then $h_*(\Gamma) = h(\Gamma)$; if $(\Gamma, x) \in \overline{\mathcal{W}(p, q)}$ (or $\overline{\mathcal{D}(p)}$), then $h_*(\Gamma, x) = (h(\Gamma), h(x))$. Clearly, h_* is a bijection and $(h_*)^{-1} = (h^{-1})_*$.

Theorem 6.7. The maps $h_* : \overline{\mathcal{M}(p, q)} \rightarrow \overline{\mathcal{M}(h(p), h(q))}$, $h_* : \overline{\mathcal{D}(p)} \rightarrow \overline{\mathcal{D}(h(p))}$, and $h_* : \overline{\mathcal{W}(p, q)} \rightarrow \overline{\mathcal{W}(h(p), h(q))}$ are homeomorphisms.

Proof. It suffices to prove that h_* is continuous because this implies h_*^{-1} is also continuous.

(1). We consider the case of $h_* : \overline{\mathcal{M}(p, q)} \rightarrow \overline{\mathcal{M}(h(p), h(q))}$.

By the definition, $\overline{\mathcal{M}(p, q)}$ is identified with a topological subspace of $\prod_{i=0}^l f_1^{-1}(a_i)$ and $\overline{\mathcal{M}(h(p), h(q))}$ is identified with a topological subspace of $\prod_{i=0}^k f_2^{-1}(b_i)$. By this identification, for any $\Gamma \in \overline{\mathcal{M}(p, q)}$, we have $\Gamma = (x_0(\Gamma), \dots, x_l(\Gamma))$ and $h_*(\Gamma) = (y_0(h(\Gamma)), \dots, y_k(h(\Gamma)))$. Suppose $x_0(\Gamma_0)$ is on $\gamma \in \mathcal{M}(p, r)$ and γ is a

component of Γ_0 , then $h(x_0(\Gamma_0))$ is on $h(\gamma) \in \mathcal{M}(h(p), h(r))$. Suppose the regular values in $[f_2(h(r)), f_2(h(p))]$ are b_0, \dots, b_s . Then $h(\gamma)$ intersects with $f_2^{-1}(b_i)$ ($0 \leq i \leq s$) at $y_i(h(\Gamma_0))$. When Γ converges to Γ_0 , we have $h(x_0(\Gamma))$ converges to $h(x_0(\Gamma_0))$. Thus, when Γ is close to Γ_0 enough, the flow line passing through $h(x_0(\Gamma))$ intersects with $f_2^{-1}(b_i)$ ($0 \leq i \leq s$) at $y_i(h(\Gamma))$ and $y_i(h(\Gamma))$ is continuous with respect to Γ .

By an induction, we can prove that, for all $0 \leq i \leq k$, $y_i(h(\Gamma))$ is continuous with respect to Γ . Thus h_* is continuous.

(2). Since $\overline{\mathcal{W}(p, q)}$ is a topological subspace of $\overline{\mathcal{M}(p, q)} \times M_1$, by (1), we infer that h_* is continuous on $\overline{\mathcal{W}(p, q)}$.

(3). We consider the case of $h_* : \overline{\mathcal{D}(p)} \rightarrow \overline{\mathcal{D}(h(p))}$.

It suffices to check the continuity of h_* on each $U(i)$. Suppose $(\Gamma_0, z_0) \in U(i)$ and $\tilde{c}_{s+1} < f_2(h(z_0)) < \tilde{c}_{s-1}$, where \tilde{c}_j are critical values of f_2 . Then $h_*(\Gamma_0, z_0) \in \tilde{U}(s)$, where $\tilde{U}(s) \subseteq \overline{\mathcal{D}(h(p))}$ is defined similarly to $U(i)$. Thus, when (Γ, z) is close to (Γ_0, z_0) enough, we have $h_*(\Gamma, z) \in \tilde{U}(s)$. Identify $\tilde{U}(s)$ with a topological subspace of $\prod_{j=0}^{s-1} f_2^{-1}(b_j) \times M_2$, we have $h_*(\Gamma, z) = (y_0(h(\Gamma)), \dots, y_{s-1}(h(\Gamma)), h(z))$. By an argument similar to that in (1), we can prove that $y_j(h(\Gamma))$ is continuous with respect to Γ . Since $h(z)$ is continuous with respect to z , we infer h_* is continuous. \square

7. PROPERTIES OF MODULI SPACES

In this section, we establish the relevant properties of the compactified moduli spaces. Particularly, the manifold structures of these spaces will be emphasized.

When the metric is locally trivial, similar results can be found in the literature (see e.g. [13], [3] and [21]). Our results are extensions of those results to the case of a general metric provided that the Morse function f is proper. In this case, every negative gradient-like vector field X for f satisfies the *CF* condition in [21, def. 2.6]. This extension needs Theorem 4.1, Lemma 5.1, and Theorem 6.7.

We introduce the concepts of manifolds with corners or faces. Our terminology follows that in [6, p. 2], [10, sec. 1.1] and [21].

Definition 7.1. A smooth manifold with corners is a space defined in the same way as a smooth manifold except that its atlases are open subsets of $[0, +\infty)^n$.

If L is a smooth manifold with corners, $x \in L$, a neighborhood of x is diffeomorphic to $(0, \epsilon)^{n-k} \times [0, \epsilon)^k$, then define $c(x) = k$. Clearly, $c(x)$ does not depend on the choice of atlas.

Definition 7.2. Suppose L is a smooth manifold. We call $\{x \in L \mid c(x) = k\}$ the k -stratum of L . Denote it by $\partial^k L$.

Clearly, $\partial^k L$ is a submanifold *without* corners inside L , its codimension is k .

Definition 7.3. A smooth manifold L with faces is a smooth manifold with corners such that each x belongs to the closures of $c(x)$ different components of $\partial^1 L$.

Consider first the special case when M is compact. By Theorem 4.1, we can construct a negative gradient-like field Y for f such that Y is locally trivial and satisfies transversality. In addition, there exists a topological equivalence between X and Y such that $h(p) = p$ for each critical point p . Thus, by Theorem 6.7, X and Y have isomorphic compactified moduli spaces. Since the properties of these

spaces for Y are proved in [21]. We deduce certain properties of these spaces for X .

More generally, suppose that f is proper but M is not necessarily compact. For any pair of critical points (p, q) , choose regular values a and b such that $M^{a,b}$ is compact and contains p and q . By Lemma 5.1, we can embed $M^{a,b}$ into \widetilde{M} , extend $f|_{M^{a,b}}$ to be \widetilde{f} on \widetilde{M} , and extend $X|_{M^{a,b}}$ to be \widetilde{X} on \widetilde{M} . Furthermore, $\overline{\mathcal{W}(p, q; X)} = \overline{\mathcal{W}(p, q; \widetilde{X})}$. Thus we get $\overline{\mathcal{M}(p, q; X)} = \overline{\mathcal{M}(p, q; \widetilde{X})}$ and $\overline{\mathcal{W}(p, q; X)} = \overline{\mathcal{W}(p, q; \widetilde{X})}$. If f is bounded below, we choose M^a such that $p \in M^a$. Do the above extension again to get $\overline{\mathcal{D}(p; X)} = \overline{\mathcal{D}(p; \widetilde{X})}$. Thus Lemma 5.1 reduces the proper case to the compact case.

Before formulating the property of $\overline{\mathcal{M}(p, q)}$, we introduce a map. Suppose $\Gamma_1 \in \overline{\mathcal{M}(p, r)}$ is a generalized flow line connecting p with r and $\Gamma_2 \in \overline{\mathcal{M}(r, q)}$ is a generalized flow line connecting r with q . Thus the combination of Γ_1 and Γ_2 gives a generalized flow line Γ connecting p with q . So we have the natural inclusion $i_{(p,r,q)} : \overline{\mathcal{M}(p, r)} \times \overline{\mathcal{M}(r, q)} \rightarrow \overline{\mathcal{M}(p, q)}$.

Theorem 7.4. *Suppose f is proper and X satisfies transversality. Then, for each pair of critical points (p, q) , the space $\overline{\mathcal{M}(p, q)} = \bigsqcup \mathcal{M}_I$ is defined as Definition 6.4. It has the flowing properties.*

- (1). *It is a compact topological manifold with boundary. Its interior is $\mathcal{M}(p, q)$.*
- (2). *Its topology is compatible with those of \mathcal{M}_I , and the map $i_{(p,r,q)} : \overline{\mathcal{M}(p, r)} \times \overline{\mathcal{M}(r, q)} \rightarrow \overline{\mathcal{M}(p, q)}$ is a topological embedding.*
- (3). *The evaluation map $E : \overline{\mathcal{M}(p, q)} \rightarrow \prod_{i=0}^l f^{-1}(a_i)$ is a topological embedding, where E is defined in (6.2).*
- (4). *There exists a topological embedding $\iota : \overline{\mathcal{M}(p, q)} \rightarrow \prod_{i=0}^l f^{-1}(a_i)$ such that $\iota(\overline{\mathcal{M}(p, q)})$ is a smoothly embedded submanifold with faces inside $\prod_{i=0}^l f^{-1}(a_i)$ and the k -stratum of $\iota(\overline{\mathcal{M}(p, q)})$ is $\bigsqcup_{|I|=k} \iota(\mathcal{M}_I)$.*

In particular, if M is compact, then there exist homeomorphisms $h_i : f^{-1}(a_i) \rightarrow f^{-1}(a_i)$ such that $\iota = (\prod_{i=0}^l h_i) \circ E$ in (4).

Theorem 7.5. *Under the assumption of Theorem 7.4, each $\overline{\mathcal{M}(p, q)}$ carries a smooth structure compatible with its topology such that $\overline{\mathcal{M}(p, q)}$ is a compact smooth manifold with faces and $\partial^k \overline{\mathcal{M}(p, q)} = \bigsqcup_{|I|=k} \mathcal{M}_I$. In particular, suppose M is compact, then $i_{(p,r,q)} : \overline{\mathcal{M}(p, r)} \times \overline{\mathcal{M}(r, q)} \rightarrow \overline{\mathcal{M}(p, q)}$ is a smooth embedding.*

Remark 7.1. The (1) of Theorem 7.4 shows that we can add a boundary to $\mathcal{M}(p, q)$ such that it becomes a compact manifold with boundary. The following theorems show that this is also true for $\mathcal{W}(p, q)$ and $\mathcal{D}(p)$. Thus moduli spaces are *special* open manifolds (if they are open) because there exists an obstruction of adding a boundary to a general open manifold.

Remark 7.2. The paper [21, example 3.1] shows that, if the metric is not locally trivial, then $E(\overline{\mathcal{M}(p, q)})$ usually is even not a C^1 embedded submanifold of $\prod_{i=0}^l f^{-1}(a_i)$. Here E is the evaluation map in the (3) of Theorem 7.4. However, the (4) of Theorem 7.4 shows that a suitable embedding ι makes the image good.

Proof of Theorem 7.4. Choose regular values a and b such that $M^{a,b}$ is compact and contains p and q . As described in the above, construct \widetilde{M} , \widetilde{f} and \widetilde{X} . We have

$\overline{\mathcal{M}(p, q; X)} = \overline{\mathcal{M}(p, q; \tilde{X})}$ and $\mathcal{M}_I(\tilde{X}) = \mathcal{M}_I(X)$ for all critical sequences I with head p and tail q . There exists a topological equivalence $h : \tilde{M} \rightarrow \tilde{M}$ which maps the orbits of \tilde{X} to those of Y , where Y is locally trivial.

(1). By [21, thm. 3.3], we know that $\overline{\mathcal{M}(p, q; Y)}$ is a compact smooth manifold with faces whose k -stratum is $\bigsqcup_{|I|=k} \mathcal{M}_I(Y)$. Thus $\overline{\mathcal{M}(p, q; Y)}$ is a compact topological manifold with boundary, and its interior is $\mathcal{M}(p, q; Y)$. By Theorem 6.7, we know that h induces a homeomorphism $h_* : \overline{\mathcal{M}(p, q; \tilde{X})} \rightarrow \overline{\mathcal{M}(p, q; Y)}$ such that $h_*(\mathcal{M}_I(\tilde{X})) = \mathcal{M}_I(Y)$. This completes the proof of (1).

(2). The proof is easy and even does not need the comparison among $\overline{\mathcal{M}(p, q; X)}$, $\overline{\mathcal{M}(p, q; \tilde{X})}$ and $\overline{\mathcal{M}(p, q; Y)}$. Similar details is also included in the proof of [21, thm. 3.3].

(3). This is the definition of the topology of $\overline{\mathcal{M}(p, q; X)}$.

(4). Let $E_Y : \overline{\mathcal{M}(p, q; Y)} \rightarrow \prod_{i=0}^l \tilde{f}^{-1}(a_i)$ be the evaluation map. By [21, thm. 3.3], we know E_Y is a smooth embedding. We shall prove that $\text{Im}(E_Y) \subseteq \prod_{i=0}^l f^{-1}(a_i) \subseteq \prod_{i=0}^l \tilde{f}^{-1}(a_i)$. It suffices to prove that $\mathcal{W}(r_1, r_2; Y) \subseteq M^{a,b}$ for all r_1 and r_2 in $M^{a,b}$.

Suppose γ is a flow line in \tilde{M} such that $\gamma(t_0) \in M^{a,b}$ and $\gamma(t_1) \notin M^{a,b}$ for some t_0 and t_1 . Then either $\tilde{f}(\gamma(t_1)) > b > f(r_1)$ or $\tilde{f}(\gamma(t_1)) < a < f(r_2)$. Thus $\mathcal{W}(r_1, r_2; Y) \subseteq M^{a,b}$.

Thus $\text{Im}(E_Y) \subseteq \prod_{i=0}^l f^{-1}(a_i)$ and $\iota = E_Y \circ h_*$ is the desired map.

Finally, we consider the special case when M is compact.

We construct Y on M . The topological equivalence $h : M \rightarrow M$ induces the homeomorphism $h_* : \overline{\mathcal{M}(p, q; X)} \rightarrow \overline{\mathcal{M}(p, q; Y)}$. We consider the relation between $h(f^{-1}(a_i))$ and $f^{-1}(a_i)$. Denote by ϕ_t^1 the flow generated by X and by ϕ_t^2 the flow generated by Y . For any $x \in f^{-1}(a_i)$, we have $\phi^1(-\infty, x) = r_1$ for some $r_1 \in M - M^{a_i}$ and $\phi^1(+\infty, x) = r_2$ for some $r_2 \in M^{a_i}$. Since h is a topological equivalence fixing r_1 and r_2 , we know that $\phi^2(-\infty, h(x)) = r_1$ and $\phi^2(+\infty, h(x)) = r_2$. Thus $\phi^2(t, h(x)) \in f^{-1}(a_i)$ for some $t \in (-\infty, +\infty)$. An isotopy along the flows generated by Y gives a homeomorphism $\psi_i : h(f^{-1}(a_i)) \rightarrow f^{-1}(a_i)$. We complete the proof by defining $h_i = \psi_i \circ h$. \square

The first half part of Theorem 7.5 is a corollary of Theorem 7.4. We can construct the topological equivalence on M when M is compact. Thus the second half part is also true because it is true in the special case.

Since we define $\overline{\mathcal{W}(p, r)}$ as a subspace of $\overline{\mathcal{M}(r, q)} \times M$, we have the inclusion $i : \overline{\mathcal{W}(p, r)} \rightarrow \overline{\mathcal{M}(r, q)} \times M$. If $(\Gamma_1, x) \in \overline{\mathcal{W}(p, r)}$ and $\Gamma_2 \in \overline{\mathcal{M}(r, q)}$, then the combination of Γ_1 and Γ_2 gives an element in $\overline{\mathcal{M}(p, q)}$ and x is on it. This defines a natural inclusion $i_{(p,r,q)}^1 : \overline{\mathcal{W}(p, r)} \times \overline{\mathcal{M}(r, q)} \rightarrow \overline{\mathcal{W}(p, q)}$. Similarly, we can define a natural inclusion $i_{(p,r,q)}^2 : \overline{\mathcal{M}(p, r)} \times \overline{\mathcal{W}(r, q)} \rightarrow \overline{\mathcal{W}(p, q)}$.

Suppose f is bounded below. We define the evaluation map $e : \overline{\mathcal{D}(p)} \rightarrow M$ as $e(\Gamma, x) = x$. Clearly, the restriction of e on $\mathcal{D}_I = \mathcal{M}_I \times \overline{\mathcal{D}(r_k)}$ is the coordinate projection onto $\overline{\mathcal{D}(r_k)} \subseteq M$. If $\Gamma_1 \in \overline{\mathcal{M}(p, r)}$ and $(\Gamma_2, x) \in \overline{\mathcal{D}(r)}$, then the combination of Γ_1 and Γ_2 is a generalized flow line connecting p with x . This defines a natural inclusion $i_{(p,r)} : \overline{\mathcal{M}(p, r)} \times \overline{\mathcal{D}(r)} \rightarrow \overline{\mathcal{D}(p)}$.

By [21, thms. 3.4, 3.5 and 3.7], using an argument similar to the proof of Theorem 7.4, we can get the following results. The proof of Theorem 7.6 needs the fact that

the map \tilde{E} defined in (6.4) is a smooth embedding when the vector field X is locally trivial. Although this fact is not stated in [21], its easy to see that it is true from the proof of [21, thm. 3.5].

Theorem 7.6. *Suppose f is proper and X satisfies transversality. Then, for each pair of critical points (p, q) , the space $\overline{\mathcal{W}(p, q)} = \bigsqcup_{(I, s)} \mathcal{W}_{I, s}$ is defined as Definition 6.5. It has the flowing properties.*

- (1). *It is a compact topological manifold with boundary. Its interior is $\mathcal{W}(p, q)$.*
- (2). *Its topology is compatible with that of $\mathcal{W}_{I, s}$. The maps $i_{(p, r, q)}^1 : \overline{\mathcal{W}(p, r)} \times \overline{\mathcal{M}(r, q)} \rightarrow \overline{\mathcal{W}(p, q)}$ and $i_{(p, r, q)}^2 : \overline{\mathcal{M}(p, r)} \times \overline{\mathcal{W}(r, q)} \rightarrow \overline{\mathcal{W}(p, q)}$ are topological embeddings.*
- (3). *The inclusion $i : \overline{\mathcal{W}(p, r)} \rightarrow \overline{\mathcal{M}(r, q)} \times M$ and the map $\tilde{E} : \overline{\mathcal{W}(p, q)} \rightarrow \prod_{i=0}^l f^{-1}(a_i) \times M$ are topological embeddings, where \tilde{E} is defined in (6.4).*
- (4). *There exists a topological embedding $\iota : \overline{\mathcal{W}(p, q)} \rightarrow \prod_{i=0}^l f^{-1}(a_i) \times M$ such that $\iota(\overline{\mathcal{W}(p, q)})$ is a smoothly embedded submanifold with faces inside $\prod_{i=0}^l f^{-1}(a_i) \times M$ and the k -stratum of $\iota(\overline{\mathcal{W}(p, q)})$ is $\bigsqcup_{(I, s)} \iota(\mathcal{W}_{I, s})$, where (I, s) contains $k + 2$ components.*

In particular, if M is compact, then there exist homeomorphisms $h_i : f^{-1}(a_i) \rightarrow f^{-1}(a_i)$ such that $\iota = [(\prod_{i=0}^l h_i) \times h] \circ \tilde{E}$ in (4).

Corollary 7.7. *Under the assumption of Theorem 7.6, $\overline{\mathcal{W}(p, q)}$ carries a smooth structure compatible with its topology such that $\overline{\mathcal{W}(p, q)}$ is a compact smooth manifold with faces and $\partial^k \overline{\mathcal{W}(p, q)} = \bigsqcup_{(I, s)} \mathcal{W}_{I, s}$, where (I, s) contains $k + 2$ components.*

Theorem 7.8. *Suppose f is proper and bounded below. Suppose X satisfies transversality. Then, for each critical point p , $\overline{\mathcal{D}(p)} = \bigsqcup \mathcal{D}_I$ is defined as Definition 6.6. It has the following properties.*

- (1). *It is homeomorphic to a closed disc. Its interior is $\mathcal{D}(p)$.*
- (2). *Its topology is compatible with those of \mathcal{D}_I . The map $i_{(p, r)} : \overline{\mathcal{M}(p, r)} \times \overline{\mathcal{D}(r)} \rightarrow \overline{\mathcal{D}(p)}$ is a topological embedding.*
- (3). *The evaluation map $e : \overline{\mathcal{D}(p)} \rightarrow M$ is continuous. The restriction of e on $\mathcal{D}_I = \mathcal{M}_I \times \mathcal{D}(r_k)$ is the coordinate projection onto $\mathcal{D}(r_k) \subseteq M$, where $I = \{p, r_1, \dots, r_k\}$.*
- (4). *It carries a smooth structure compatible with its topology such that it is a compact smooth manifold with faces and $\partial^k \overline{\mathcal{D}(p)} = \bigsqcup_{|I|=k-1} \mathcal{D}_I$.*

8. ORIENTATION FORMULAS

In this section, we shall prove the following orientation formulas.

Theorem 8.1 (Orientation Formulas). *Suppose f is proper and X satisfies transversality. As oriented topological manifolds, we have*

- (1). $\partial^1 \overline{\mathcal{M}(p, q)} = \bigsqcup_{p \succ r \succ q} (-1)^{\text{ind}(p) - \text{ind}(r)} \mathcal{M}(p, r) \times \mathcal{M}(r, q);$
- (2). $\partial^1 \overline{\mathcal{D}(p)} = \bigsqcup_{p \succ r} \mathcal{M}(p, r) \times \mathcal{D}(r)$, where f is bounded below;
- (3). $\partial^1 \overline{\mathcal{W}(p, q)} = \bigsqcup_{p \succeq r \succ q} (-1)^{\text{ind}(p) - \text{ind}(r) + 1} \mathcal{W}(p, r) \times \mathcal{M}(r, q) \sqcup \bigsqcup_{p \succ r \succeq q} \mathcal{M}(p, r) \times \mathcal{W}(r, q).$

In the above, $\partial^1 \square$ are equipped with boundary orientations, $\square \times \square$ are equipped with product orientations, and $\text{ind}(\ast)$ is the Morse index of \ast .

In order to explain the concepts in Theorem 8.1, we need to review the definition of orientation at first.

Suppose M is an n dimensional smooth manifold. In algebraic topology, the orientation of M at x is a generator $\alpha \in H^n(M, M - \{x\})$. In differential topology, the orientation is an ordered base $\{e_1, \dots, e_n\} \subseteq T_x M$. These two definitions are related as follows. Choose a smooth embedding $\varphi : V \rightarrow M$ such that $\varphi(0) = x$ and $D\varphi(0) = \text{Id}$, where V is a neighborhood of 0 in $T_x M$. Then $\varphi^* \alpha \in H^n(V, V - \{0\}) = H^n(T_x M, T_x M - \{0\})$ is a generator. Here $\varphi^* \alpha$ does not depend on the choice of φ . Actually, if $\tilde{\varphi}$ is another such embedding, then there exists an isotopy between φ and $\tilde{\varphi}$ in a smaller neighborhood of 0. Denote by α_0 the preferred generator in $H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ (see [15, p. 266]). The ordered base $\{e_1, \dots, e_n\}$ determines a linear isomorphism $A : T_x M \rightarrow \mathbb{R}^n$, then $A^* \alpha_0 \in H^n(T_x M, T_x M - \{0\})$ is also a generator. We say that these two definitions give the same orientation if and only if $\varphi^* \alpha = A^* \alpha_0$.

Suppose L is a k dimensional embedded submanifold of M such that its normal bundle is orientable. Choose a neighborhood U of L such that L is closed in U . Choose a Thom class $\beta \in H^{n-k}(U, U - L)$. The Thom class β defines the normal orientation in the sense of algebraic topology. On the other hand, for any $x \in L$, choose an ordered base $\{\varepsilon_{k+1}, \dots, \varepsilon_n\}$ of the normal space $N_x(L, M) = T_x M / T_x L$. This defines the normal orientation of L at x in the sense of differential topology. These two definitions are related as follows. Let $\varphi : V \rightarrow M$ be a smooth embedding such that $\varphi(0) = x$ and $P \cdot D\varphi(0) = \text{Id}$, where V is a neighborhood of 0 in $N_x(L, M)$ and $P : T_x M \rightarrow T_x M / T_x L = N_x(L, M)$ is the projection. Then $\varphi^* \beta \in H^{n-k}(V, V - \{0\}) = H^{n-k}(N_x, N_x - \{0\})$ is a generator. Here $\varphi^* \beta$ does not depend on the choice of φ . The ordered base determines an isomorphism $A : N_x \rightarrow \mathbb{R}^{n-k}$. So $A^* \alpha_0$ is also a generator of $H^{n-k}(N_x, N_x - \{0\})$, where α_0 is the preferred generator of $H^{n-k}(\mathbb{R}^{n-k}, \mathbb{R}^{n-k} - \{0\})$. These two definitions coincide if and only if $\varphi^* \beta = A^* \alpha_0$.

Suppose $\{e_1, \dots, e_k\} \subseteq T_x L$ represents the orientation of L and $\{e_{k+1}, \dots, e_n\} \subseteq T_x M$ represents the normal orientation of L . We say the orientation $\{e_1, \dots, e_n\}$ of M is defined by the orientation and the normal orientation of L . We have the following lemma whose proof is in the Appendix.

Lemma 8.2. *Suppose M_i ($i = 1, 2$) is a smooth orientable manifold, L_i is a closed orientable submanifold of M_i (which means L_i is a closed subset). Suppose the orientation and the normal orientation of L_i define the orientation of M_i . Let $\beta_i \in H^{n-k}(M_i, M_i - L_i)$ be the Thom class representing the normal orientation of L_i . Let $h : (M_1, L_1) \rightarrow (M_2, L_2)$ be a homeomorphism such that h preserves the orientation of M_i and $h^* \beta_2 = \beta_1$. Then h preserves the orientation of L_i .*

Following [21], we define the orientations of $\mathcal{D}(p)$, $\mathcal{W}(p, q)$ and $\mathcal{M}(p, q)$. We review the definition by means of differential topology in [21, p. 500] as follows (see [21] for more details).

Assign an arbitrary orientation to $\mathcal{D}(p)$ for each critical point p . Since $\mathcal{D}(q)$ and $\mathcal{A}(q)$ are transversal, the orientation of $\mathcal{D}(q)$ gives the normal bundle $N(\mathcal{A}(q), M) = T_{\mathcal{A}(q)} M / T\mathcal{A}(q)$ an orientation. Since $\mathcal{D}(p)$ is transverse to $\mathcal{A}(q)$ and $\mathcal{W}(p, q) = \mathcal{D}(p) \cap \mathcal{A}(q)$, the orientation of $N(\mathcal{A}(q), M)$ gives the normal bundle $N(\mathcal{W}(p, q), \mathcal{D}(p))$

an orientation. We choose the orientation of $\mathcal{W}(p, q)$ such that the orientation and the normal orientation of $\mathcal{W}(p, q)$ define the orientation of $\mathcal{D}(p)$. Identify $\mathcal{M}(p, q)$ with $\mathcal{W}(p, q) \cap f^{-1}(a)$ for some regular value $a \in (f(q), f(p))$. The orientation of $\mathcal{W}(p, q) \cap f^{-1}(a)$ is defined by the direction of the flow and the orientation of $\mathcal{W}(p, q)$. This defines the orientation of $\mathcal{M}(p, q)$. This definition does not depend on the choice of a .

By Theorems 7.4, we know $\overline{\mathcal{M}(p, q)}$ is a topological manifold with boundary, whose interior is $\mathcal{M}(p, q)$. Thus the orientation of $\mathcal{M}(p, q)$ gives $\partial\overline{\mathcal{M}(p, q)}$ the **boundary orientation** in the usual sense. In other words, the combination of the outward normal direction and the boundary orientation of the boundary gives the orientation of the manifold. Also by Theorem 7.4, we know that $\partial^1\overline{\mathcal{M}(p, q)} = \bigsqcup_{|I|=1} \mathcal{M}_I = \bigsqcup_{p \succ r \succ q} \mathcal{M}(p, r) \times \mathcal{M}(r, q)$ is an open subset of $\partial\overline{\mathcal{M}(p, q)}$. Thus $\partial^1\overline{\mathcal{M}(p, q)}$ has the boundary orientation. On the other hand, both $\mathcal{M}(p, r)$ and $\mathcal{M}(r, q)$ have orientations. Thus $\mathcal{M}(p, r) \times \mathcal{M}(r, q)$ has the **product orientation**. We shall consider the relation between these two orientations. Similarly, $\mathcal{D}(p)$ and $\mathcal{M}(p, q)$ also have such issues. Theorem 8.1 indicates these relations.

Similarly to the previous section, by Lemma 5.1, we may assume that M is compact. By Theorem 4.1, we can construct the locally trivial field Y and the topological equivalence h mapping the orbits of X to those of Y .

However, since h is not assumed differentiable, we have to use the algebraic method to describe the orientation of $\mathcal{W}(p, q; X)$ again. Choose an open tubular neighborhood U_q of $\mathcal{A}(q; X)$ such that $\mathcal{A}(q; X)$ is closed in U_q . Suppose the index $\text{ind}(q) = s$. We have the inclusion isomorphism

$$H^s(U_q, U_q - \mathcal{A}(q; X)) \xrightarrow{\cong} H^s(U_q \cap \mathcal{D}(q; X), U_q \cap \mathcal{D}(q; X) - \{q\}),$$

where $H^s(U_q \cap \mathcal{D}(q; X), U_q \cap \mathcal{D}(q; X) - \{q\}) = H^s(\mathcal{D}(q; X), \mathcal{D}(q; X) - \{q\})$. Thus the orientation of $\mathcal{D}(q; X)$, $\alpha_q \in H^s(\mathcal{D}(q; X), \mathcal{D}(q; X) - \{q\})$, determines a Thom class $\beta_q \in H^s(U_q, U_q - \mathcal{A}(q; X))$. Let $U_{p,q} = \mathcal{D}(p; X) \cap U_q$. Then $U_{p,q}$ is open in $\mathcal{D}(p; X)$ and $\mathcal{W}(p, q; X)$ is closed in $U_{p,q}$. By the inclusion monomorphism (it is an isomorphism if and only if $U_{p,q}$ is connected)

$$H^s(U_q, U_q - \mathcal{A}(q; X)) \longrightarrow H^s(U_{p,q}, U_{p,q} - \mathcal{W}(p, q; X)),$$

we have that β_q determines a Thom class $\beta_{p,q} \in H^s(U_{p,q}, U_{p,q} - \mathcal{W}(p, q; X))$. Clearly, $U_{p,q}$ inherits the orientation from $\mathcal{D}(p; X)$. Thus $\beta_{p,q}$ and the orientation of $U_{p,q}$ give $\mathcal{W}(p, q; X)$ the orientation.

Since $h(\mathcal{D}(p; X)) = \mathcal{D}(p; Y)$, we can define the orientation of $\mathcal{D}(p; Y)$ as $\alpha'_p = (h^{-1})^* \alpha_p \in H^s(\mathcal{D}(p; Y), \mathcal{D}(p; Y) - \{p\})$ for each p . Then the orientations of $\mathcal{W}(p, q; Y)$ are defined. We also have $h(\mathcal{W}(p, q; X)) = \mathcal{W}(p, q; Y)$.

Lemma 8.3. *The topological equivalence h preserves the orientation of $\mathcal{W}(p, q; X)$.*

Proof. Choose the open tubular neighborhood U_q of $\mathcal{A}(q; X)$ and define $U_{p,q} = \mathcal{D}(p; X) \cap U_q$ as the above. Define $U'_q = h(U_q)$ and $U'_{p,q} = h(U_{p,q})$. We may assume $U_{p,q}$ is connected.

Suppose the orientation of $\mathcal{D}(q; Y)$ defines the Thom class $\beta'_q \in H^s(U'_q, U'_q - \mathcal{A}(q; Y))$ and the Thom class $\beta'_{p,q} \in H^s(U'_{p,q}, U'_{p,q} - \mathcal{W}(p, q; Y))$. We have the

following commutative diagram.

$$\begin{array}{ccc}
H^s(U'_{p,q}, U'_{p,q} - \mathcal{W}(p, q; Y)) & \xrightarrow{h^*} & H^s(U_{p,q}, U_{p,q} - \mathcal{W}(p, q; X)) \\
\uparrow & & \uparrow \\
H^s(U'_q, U'_q - \mathcal{A}(q; Y)) & \xrightarrow{h^*} & H^s(U_q, U_q - \mathcal{A}(q; X)) \\
\downarrow & & \downarrow \\
H^s(U'_q \cap \mathcal{D}(q; Y), U'_q \cap \mathcal{D}(q; Y) - \{q\}) & \xrightarrow{h^*} & H^s(U_q \cap \mathcal{D}(q; X), U_q \cap \mathcal{D}(q; X) - \{q\})
\end{array}$$

All of these maps are isomorphisms. The vertical maps are induced by inclusions. Since $h^* \alpha'_q = \alpha_q$, we have $h^* \beta'_q = \beta_q$. Thus we get $h^* \beta'_{p,q} = \beta_{p,q}$.

We also know that h preserves the orientation of $U_{p,q}$. By Lemma 8.2, the proof is completed. \square

As in Theorem 6.7, let $h_* : \overline{\mathcal{M}(p, q; X)} \rightarrow \overline{\mathcal{M}(p, q; Y)}$, $h_* : \overline{\mathcal{W}(p, q; X)} \rightarrow \overline{\mathcal{W}(p, q; Y)}$ and $h_* : \overline{\mathcal{D}(p; X)} \rightarrow \overline{\mathcal{M}(p; Y)}$ be the maps induced by h . Since h preserves the direction of flow, by Lemma 8.3, we get the following immediately.

Lemma 8.4. *The map h_* preserves the orientation of $\mathcal{M}(p, q; X)$.*

Proof of Theorem 8.1. Consider the map h_* defined in the above. Clearly, h_* is identical to h on $\mathcal{D}(p; X)$ and $\mathcal{W}(p, q; X)$.

By the definition of the orientation of $\mathcal{D}(p; Y)$, we know h_* preserves the orientation of $\mathcal{D}(p; X)$. Combining this fact with Lemmas 8.3 and 8.4, we infer that h_* preserves both the boundary orientations and the product orientations. Thus h_* preserves the orientation relations. Since these formulas for Y are proved in [21, thm. 3.6], we infer that the orientation formulas are valid for X . \square

9. CW STRUCTURES

In this section, We shall prove the following three theorems on the CW structures.

Theorem 9.1. *Suppose f is proper and bounded below. Suppose X satisfies transversality. Suppose a is a regular value of f . Define $K^a = \bigsqcup_{f(p) \leq a} \mathcal{D}(p)$ with the topology induced from M . Then K^a is a finite CW complex with characteristic maps $e : \overline{\mathcal{D}(p)} \rightarrow K^a$, where e is defined in (3) of Theorem 7.8. The inclusion $K^a \hookrightarrow M^a$ is a simple homotopy equivalence. In fact, there is a CW decomposition of M^a such that K^a expands to M^a by elementary expansions.*

Theorem 9.2. *Under the assumption of Theorem 9.1, define $K = \bigsqcup_{p \in M} \mathcal{D}(p)$. Define the topology of K as the direct limit of that of K^a when a tends to $+\infty$. Then K is a countable CW complex with characteristic maps $e : \overline{\mathcal{D}(p)} \rightarrow K$, where e is defined in (3) of Theorem 7.8. Furthermore, the inclusion $i : K \hookrightarrow M$ is a homotopy equivalence.*

As mentioned before, $\dim(\mathcal{M}(p, q)) = \text{ind}(p) - \text{ind}(q) - 1$. If $\text{ind}(q) = \text{ind}(p) - 1$, then $\mathcal{M}(p, q)$ is a 0 dimensional manifold. Actually, $\mathcal{M}(p, q)$ consists of finitely many points because it is compact in this case.

Theorem 9.3. *Let K^a (or K) be the CW complex in Theorem 9.1 (or 9.2). Let $C_*(K^a)$ (or $C_*(K)$) be the associated cellular chain complex and $[\overline{\mathcal{D}(p)}]$ be the base element represented by the oriented $\overline{\mathcal{D}(p)}$ in $C_*(K^a)$ (or $C_*(K)$). Then*

$$\partial[\overline{\mathcal{D}(p)}] = \sum_{\text{ind}(q)=\text{ind}(p)-1} \# \mathcal{M}(p, q) [\overline{\mathcal{D}(q)}],$$

where $\# \mathcal{M}(p, q)$ is the sum of the orientations ± 1 of all points in $\mathcal{M}(p, q)$ defined in Theorem 8.1, and $\text{ind}(\ast)$ is the Morse index of \ast .

Remark 9.1. Consider the special case when M is compact. Theorem 9.1 shows that the compactified descending manifolds give a bona fide CW decomposition of M . Before the invention of the theory of Moduli spaces, this problem was addressed in [11, thm. 1] and [12, rem. 3], which show the existence of the characteristic maps under the assumption that the vector field is locally trivial. Besides the simple homotopy type, Theorem 9.1 strengthens their solution in two ways. Firstly, the characteristic maps here $e : \overline{\mathcal{D}(p)} \rightarrow M$ have the explicit formula defined in (3) of Theorem 7.8. Secondly, we drop the assumption of the local triviality of the vector field. In the case when f has only one critical point of index 0, the paper [1, lem. 2.15] also gives a answer similar to Theorem 9.1.

Remark 9.2. The above theorems show that $C_*(K)$ computes the homology of M , and its boundary operator ∂ coincides with that of Morse homology. This shows Morse homology arises from a cellular chain complex. For Morse homology, see [14, cor. 7.3] and [22].

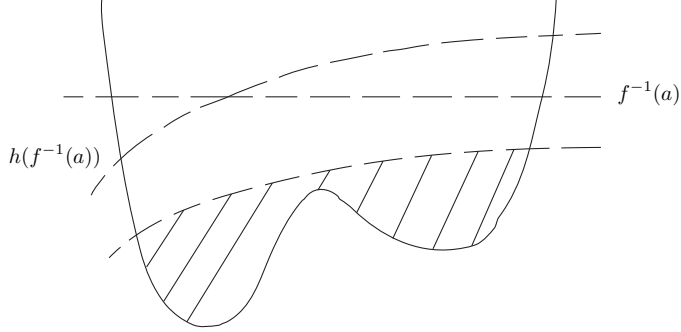
Proof of Theorem 9.1. By Theorem 7.8, $\overline{\mathcal{D}(p)}$ is a closed disc and e is continuous. Thus K^a is a finite CW complex with characteristic maps e .

We shall construct the desired CW decomposition of M^a .

Suppose M is not compact. By Lemma 5.1, we can embed M^a into \widetilde{M} and extend $f|_{M^a}$ to be \tilde{f} on \widetilde{M} such that $\tilde{f}|_{\widetilde{M}-M^a} > a$. We get $\widetilde{M}^a = M^a$. As a result, we may assume M is compact.

By Theorem 4.1, we can construct a locally trivial field Y on M and a topological equivalence h which maps the orbits of Y to those of X . Clearly, $h(\mathcal{D}(p; Y)) = \mathcal{D}(p; X)$ and $h(K^a(Y)) = K^a$ where $K^a(Y) = \bigsqcup_{f(p) \leq a} \mathcal{D}(p; Y)$. By [21, thm. 3.8], there exists a CW decomposition of M^a such that $K^a(Y)$ expands to M^a by elementary expansions. Thus it suffices to prove that there exists a homeomorphism $\tilde{h} : M^a \rightarrow M^a$ such that \tilde{h} and h coincide on $K^a(Y)$.

Denote by ϕ_t^1 the flow generated by X and by ϕ_t^2 the flow generated by Y . For any $x \in f^{-1}(a)$, we have $\phi^2(-\infty, x) = r_1$ for some $r_1 \in M - M^a$ and $\phi^2(+\infty, x) = r_2$ for some $r_2 \in M^a$. Since h is a topological equivalence fixing r_1 and r_2 , we have $\phi_t^1(h(x))$ is a flow line between r_1 and r_2 . Thus, for any $x \in h(f^{-1}(a))$, $\phi^1(t(x), x) \in f^{-1}(a)$ for some $t(x)$ and $t(x)$ is continuous on $h(f^{-1}(a))$. Since $h(f^{-1}(a))$ is compact, there exists $T > 0$ such that $T > -t(x)$ for all $x \in h(f^{-1}(a))$. As a result, $\phi_T^1(M^a) \subseteq \text{Int}[h(M^a)]$. (This is illustrated by Figure 4, $\phi_T^1(M^a)$ is the shadowed part, M^a is the part below $f^{-1}(a)$ and $h(M^a)$ is the part below $h(f^{-1}(a))$.) By an isotopy along the flows generated by X , we can construct a homeomorphism $\psi : h(M^a) \rightarrow M^a$ such that $\psi|_{\phi_T^1(M^a)} = \text{Id}$ and $\psi(h(M^a) - \phi_T^1(M^a)) = M^a - \phi_T^1(M^a)$. Then $\tilde{h} = \psi \circ h$ is the desired homeomorphism. \square

FIGURE 4. Construction of ψ

Proof of Theorem 9.2. The CW structure of K is obvious.

By Theorem 9.1, $i : K^a \hookrightarrow M^a$ is a homotopy equivalence for any regular value a . Thus, it's straightforward to check that $i : K \hookrightarrow M$ is a weak homotopy equivalence, i.e. i induces the isomorphisms between homotopy groups. Since M carries a triangulation, by Whitehead's Theorem, i is a homotopy equivalence. \square

Proof of Theorem 9.3. There are two proofs.

First, duplicate the proof of [21, thm. 3.9]. Certainly, the local triviality of the vector field X is assumed in [21]. However, the only reason for making this assumption is that the (2) of Theorem 8.1 was proved under this assumption in [21]. In this paper, this orientation formula is true even if we drop this assumption. Thus, the first proof is valid.

Second, reduce it to the case of a locally trivial vector field Y . The map h_* in Theorem 6.7 induces an isomorphism between $C_*(K^a(X))$ and $C_*(K^a(Y))$. By Lemma 8.4, h_* preserves the orientation of $\mathcal{M}(p, q; X)$. Since this statement is true for $C_*(K^a(Y))$, the second proof is complete. \square

APPENDIX A.

In this appendix, we shall prove Lemma 8.2.

Suppose M is an n dimensional manifold. Suppose L is a connected and closed k dimensional submanifold of M . Let U be a closed tubular neighborhood of L such that U is diffeomorphic to a *closed* disk bundle over L via the exponential map. Let $i : L \hookrightarrow U$ be the inclusion and $\pi : U \rightarrow L$ be the smooth projection. Clearly, i and π are proper. Thus $\pi^* : H_C^k(L) \rightarrow H_C^k(U)$ and $i^* : H_C^k(U) \rightarrow H_C^k(L)$ are isomorphisms and they are a pair of inverses, where H_C^k is the cohomology with compact support. Furthermore, $H_C^k(L) \cong \mathbb{Z}$, its generator is an orientation of L .

Define $H_C^n(U, U - L) = \varinjlim_{K \subset L} H^n(U, U - K)$, where K is compact. We can prove the inclusion $H^n(U, U - \{x\}) \rightarrow H_C^n(U, U - L)$ is an isomorphism for any $x \in L$.

Suppose $\alpha \in H_C^k(L)$ and the Thom class $\beta \in H^{n-k}(U, U - L)$ represent the orientation and the normal orientation of L respectively. Suppose the orientation and the normal orientation define the orientation of M .

Lemma A.1. *The following cup product homomorphism is an isomorphism.*

$$H_C^k(U) \otimes H^{n-k}(U, U - L) \xrightarrow[\cong]{\cup} H_C^n(U, U - L).$$

Furthermore, via the isomorphism $H^n(U, U - \{x\}) \rightarrow H_C^n(U, U - L)$, we get $\pi^*\alpha \cup \beta \in H_C^n(U, U - L)$ represents the orientation of M in $H^n(U, U - \{x\})$.

Proof. For any $x \in L$, we have a commutative diagram

$$\begin{array}{ccc} H^k(U, U - \pi^{-1}(x)) \otimes H^{n-k}(U, U - L) & \longrightarrow & H^n(U, U - \{x\}) \\ \cong \downarrow & & \downarrow \cong \\ H_C^k(U) \otimes H^{n-k}(U, U - L) & \longrightarrow & H_C^n(U, U - L). \end{array}$$

Here the vertical maps are induced by inclusions, and the horizontal ones are given by cup product pairings. By excision and the basic property of Thom class, we can localize the argument near x . However, the disk bundle near x has a product structure. Now use the Künneth Formula to complete the proof. \square

Proof of Lemma 8.2. It suffices to prove the special case of that L_i is connected.

Let U_2 be a *closed* tubular neighborhood of L_2 with the smooth projection $\pi_2 : U_2 \rightarrow L_2$. Let $\alpha_2 \in H_C^k(L_2)$ be the orientation of L_2 , by the above lemma, we have $\pi_2^*\alpha_2 \cup \beta_2|_{U_2} = \gamma_2 \in H_C^n(U_2, U_2 - L_2)$ represents the orientation of M_2 on L_2 . Here $\beta_2|_{U_2}$ is the image of β_2 under the inclusion $H^{n-k}(M_2, M_2 - L_2) \rightarrow H^{n-k}(U_2, U_2 - L_2)$. It is the restriction of β_2 on U_2 .

Let $U'_1 = h^{-1}(U_2)$. Choose a *closed* tubular neighborhood U_1 of L_1 such that $U_1 \subseteq \text{Int}U'_1$ and $\pi_1 : U_1 \rightarrow L_1$ is a smooth projection. By the above lemma again, we have the following isomorphism

$$H_C^k(U_1) \otimes H^{n-k}(U_1, U_1 - L_1) \xrightarrow[\cong]{\cup} H_C^n(U_1, U_1 - L_1),$$

and

$$(A.1) \quad \pi_1^*\alpha_1 \cup \beta_1|_{U_1} = \gamma_1$$

represents the orientation of M_1 on L_1 , where $\beta_1|_{U_1}$ is the restriction of β_1 on U_1 .

Consider the following commutative diagram:

$$\begin{array}{ccccc} H_C^k(U_2) & \xrightarrow{h^*} & H_C^k(U'_1) & \xrightarrow{\iota^*} & H_C^k(U_1) \\ i_2^* \downarrow & & j^* \downarrow & \swarrow i_1^* & \\ H_C^k(L_2) & \xrightarrow{h^*} & H_C^k(L_1), & & \end{array}$$

where, i_1, i_2, j and ι are inclusions. Since $h^*\pi_2^*\alpha_2 \cup h^*\beta_2|_{U_2} = h^*\gamma_2$, we have $\iota^*h^*\pi_2^*\alpha_2 \cup \iota^*h^*\beta_2|_{U_2} = \iota^*h^*\gamma_2$. Since h preserves the orientation of M_1 and the Thom class, we have $\iota^*h^*\gamma_2 = \gamma_1$ and $\iota^*h^*\beta_2|_{U_2} = \beta_1|_{U_1}$. Thus

$$(A.2) \quad \iota^*h^*\pi_2^*\alpha_2 \cup \beta_1|_{U_1} = \gamma_1.$$

Since the cup product pairing above is an isomorphism, by (A.1) and (A.2), we infer $\iota^*h^*\pi_2^*\alpha_2 = \pi_1^*\alpha_1$. So we have

$$\alpha_1 = i_1^*\pi_1^*\alpha_1 = i_1^*\iota^*h^*\pi_2^*\alpha_2 = h^*i_2^*\pi_2^*\alpha_2 = h^*\alpha_2.$$

This completes the proof. \square

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